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A DUAL-SYMMETRIC DEFINITION OF FIELD.*

By Alfred L. Foster and B. A. Bernstein.

---

1. Introduction. In another paper [1][1] it was shown how the general theory of (commutative) rings (with unit) may be developed in terms of operations symmetrically related to each other, as opposed to the traditional asymmetrical $+, \times$ treatment. It was shown that this symmetrical approach discloses a *duality theorem for rings* which, in particular, subsumes the familiar duality theorem of Boolean Algebra. In this dual-symmetric approach to rings initiated in [1] the following *dual pairs*; $+, \dot{+}$; $\times, \Delta$; as well as the (self-dual) *ring complement*, $*$, were shown to play a central role, where

$a + b =$ *ordinary ring addition.*
$a \dot{+} b = a + b - 1.$
$a \times b =$ *ordinary ring multiplication (also written* ab*).*
$a \Delta b = a + b - ab$*, called "ring* Δ*-multiplication."*

It is sufficient here to recall a considerably curtailed version of the ring-duality theorem given in [1]. Let $a, b \cdots$ denote variable ring elements.

Ring-Duality Theorem. *If* $P(a, b, \cdots; +, \dot{+}; \times, \Delta; *; 0, 1)$ *is a true ring proposition, so also is the dual proposition, dl* P*, obtained by replacing*

$+$ *by* $\dot{+}$
$\dot{+}$ *by* $+$
$\times$ *by* Δ
Δ *by* $\times$
0 *by* 1
1 *by* 0
$*$ *by* $*$ (*i. e., unchanged*).

---

* Received May 8, 1944. Presented to the American Mathematical Society, March, 1944.

[1] "Symmetric approach to commutative rings, with duality theorem; Boolean duality as special case," by A. L. Foster and B. A. Bernstein, *Duke Mathematical Journal*, vol. 11 (1944), pp. 603-616.

The present paper, the second to deal with the development of the theory of rings from the dual-symmetric approach, may be read with only the above mentioned knowledge of [1]. We here give a symmetric definition of the concept of (commutative) field entirely in terms of its *dual multiplications* $\times$ and Δ. For a field (or for any species of ring) so characterized, the ring-duality theorem obviously assumes the extremely simple form:

$$dl\, P(a, b, \cdots; \times, \Delta, 0, 1) = P(a, b, \cdots; \Delta, \times, 1, 0).$$

Two sets (I and II) of postulates for the concept field are given, and moreover in each set the postulates occur in dual pairs; and therefore an independent direct proof of the duality theorem (only, of course, for fields) follows by inspection from (either set of) these postulates. In set I the definition of $+$ (in terms of the primitive notions F, $\times$, Δ, 0, 1) is given in **4**, and it is shown that with this $+$ the system $(F, +, \times)$ is (strongly) equivalent to the usual definition of field. For the most part only such theorems are proved as bear directly on this question of equivalence, though a few (of many) theorems belonging to this dual-symmetric theory have been included for their own intrinsic beauty of form. The set II is taken up in **6**.

The independence of the postulates has not been fully established, though each set, with the possible exception of the "-1 postulate" (i. e., D of set I, $=$ P7 of set II) is believed independent when appropriate restrictive clauses are introduced. In this connection it will be seen that the -1 postulate is self-dual, and hence the postulates (of either set) which remain after its deletion, (in the event of its non-independency), still occur in dual pairs.

2. The postulates, Set I. As primitive notions in the dual-symmetric definition of field to be given we take, then, an undefined class F and two undefined binary operations, $\times$ and Δ; and for convenience of formulation we also take 0 and 1 as undefined elements of F. Our primitive system is then

$$(F, \times, \Delta, 1, 0).$$

Furthermore numerous cumbersome special cases in the proofs are avoided if we assume, as we henceforth shall, that we are defining fields of at least three elements.

Notation. F_ϕ denotes the class F after deleting the element 0; similarly F_1 denotes F after deleting the element 1. Frequently $a \times b$ is abbreviated to ab. The inverse of an element a with respect to the group $\times$ (postulated below) is denoted by a', and the inverse of a with respect to the group Δ is denoted by a°. Also a'° denotes $(a')^\circ$, and similarly for $a^{\circ\prime}$, $a^{\circ\prime\circ}$, etc. Small

Roman letters a, b, etc. denote elements of F. Finally for a, b, c belonging to the group $\times$ (respectively to the group Δ), we write $(ab)c = a(bc) = abc$, (respectively $a \Delta (b \Delta c) = (a \Delta b) \Delta c = a \Delta b \Delta c$).

The postulates of set I are:

A. *0 and 1 are elements of F and F contains an element $\neq 0, \neq 1$.*

(Field) Group Postulates

B (i). *$(F_\phi, \times)$ is an Abelian group with unit element* 1.

(ii). $0a = a0 = 0$.

B′ (i). *(F_+, Δ) is an Abelian group with unit element* 0.

(ii). $1 \Delta a = a \Delta 1 = 1$.

(Field) Linking Postulates

C. $ab = a \Delta (a' \Delta b^\circ)'^\circ$ $\quad [a \neq 0, \neq 1;\ b \neq 1;\ a' \Delta b^\circ \neq 0]$

C′. $a \Delta b = a(a^\circ b')^{\circ\prime}$ $\quad [a \neq 0, \neq 1;\ b \neq 0;\ a^\circ b' \neq 1]$

D. $(aa^{\circ\prime}a'^\circ)^\circ = (b \Delta b'^\circ \Delta b^{\circ\prime})'$ $\quad [a \neq 0, \neq 1;\ b \neq 0, \neq 1]$.

It is easily verified that with $\Delta, \times, 0, 1$ interpreted as stated (for rings) in **1**, each of the postulates is a true proposition of the concept field $(F, +, \times, 0, 1)$ as ordinarily defined.

Concerning the use of the $=$ sign. In the postulates, and in the theorems to follow, the $=$ sign, except where otherwise explicitly stated, is used in a formula

$$S(a, b, \cdots) = T(a, b, \cdots) \tag{1}$$

always in the sense: the two sides take on the same value for all values of the arguments $a, b, \cdots$ belonging to the *domain* of the formula, i. e., for all values of $a, b, \cdots$ for which both sides are defined. Frequently this convention is augmented by an explicit statement, in square [ ] brackets, of the domain of the formula. The domain of one formula, (1), (or simply of a function $S(a, b, \cdots)$) is *included* ($\subset$) in that of another formula (respectively function) if for all values of the arguments for which the first is defined the second is also. With this use of $=$ certain elementary precautions are necessary in the derivation of theorems. We mention several frequently used obvious facts about class inclusion as applied to domains; in each case it is merely a question of establishing that the domain of a formula is not increased by the indicated transformations.

Σ1. If $S(a, b, \cdots) = T(a, b, \cdots)$ is a theorem (or postulate), and if U is a function, then each of: $S' = T'$; $S^\circ = T^\circ$; $U \times S = U \times T$; $U \Delta S = U \Delta T$ is also a theorem.

$\Sigma 2$. If $S(\cdots, b, \cdots) = T(\cdots, b, \cdots)$ is a theorem so are $S(\cdots, b', \cdots) = T(\cdots, b', \cdots)$ and $S(\cdots, b^\circ, \cdots) = T(\cdots, b^\circ, \cdots)$.

$\Sigma 3$. If the expression U'' (respectively $U^{\circ\circ}$) occurs in a theorem, and if the domain of the formula which results from the replacement of U'' by U (respectively of $U^{\circ\circ}$ by U) is included in the domain of the theorem, then the resulting formula is a theorem. The same holds with respect to the replacement of the expression $U \times U'$ by 1 (respectively of $U \Delta U^\circ$ by 0).

These Σ theorems will be employed without further mention.

Concerning Duality. As already observed the postulates are seen to occur in dual pairs: A and D each being self-dual; B, B′; C, C′, where the dual of any postulate is obtained by replacing

$$\begin{array}{ccc} \times & \text{by} & \Delta \\ \Delta & " & \times \\ 0 & " & 1 \\ 1 & " & 0. \end{array} \tag{2}$$

This of course implies that the inverses, $'$, $^\circ$ of $\times$ and Δ must be replaced

$$\begin{array}{ccc} ' & \text{by} & ^\circ \\ ^\circ & " & ' \end{array} \tag{2.1}$$

in obtaining the dual. In view of this the duality theorem (for fields) follows immediately from these postulates. Furthermore in proving any theorem we can dualize the proof and thereby obtain a proof for the dual theorem. It is therefore only necessary to prove one of a dual pair T, T' of theorems,—the other may then simply be written down by means of the exchanges (2) and (2.1).

3. General theorems. For convenient reference we give three immediate consequences of A, B, B′.

T1. (= THEOREM 1). $0 \neq 1$, *and F contains at least three elements.*

T2. $$\begin{aligned} 0 \Delta a &= a \Delta 0 = a \\ 1a &= a1 = a \\ 0a &= a0 = 0 \\ 1 \Delta a &= a \Delta 1 = 1. \end{aligned}$$

T3. $$\begin{aligned} 1' &= 1 \\ 0^\circ &= 0 \\ a'' &= a \\ a^{\circ\circ} &= a \\ a' = b' &\rightleftarrows a = b \end{aligned}$$

$$a^\circ = b^\circ \rightleftarrows a = b$$
$$a \neq 0 \to a' \text{ exists and } \neq 0$$
$$a \neq 1 \to a^\circ \quad " \quad " \neq 1$$
$$a \neq 0, \neq 1 \to a', a^\circ, a^{\circ\prime}, a'^\circ, \neq 0, \neq 1$$
$$(ab)' = a'b'$$
$$(a \Delta b)^\circ = a^\circ \Delta b^\circ$$
$$aa' = 1$$
$$a \Delta a^\circ = 0$$
$$ab = 0 \leftrightarrows a = 0 \text{ or } b = 0$$
$$a \Delta b = 1 \leftrightarrows a = 1 \text{ or } b = 1.$$

T4. $aa^{\circ\prime}a'^\circ = \mu = \text{constant} \neq 0 \quad [a \neq 0, \neq 1]$.

Proof. For $b \neq 0, \neq 1$

$$b \Delta b'^\circ \Delta b^{\circ\prime} \neq 1$$

by T3. Suppose that (*Case* 1): an element $b \neq 0, \neq 1$ exists such that

$$b \Delta b'^\circ \Delta b^{\circ\prime} = \tau \neq 0.$$

Then $\tau \neq 0, \neq 1$, whence $\tau' \neq 0, \neq 1$ (T3). Then from D, for $a \neq 0, \neq 1$

$$(aa^{\circ\prime}a'^\circ)^\circ = \tau'$$

and the theorem is proved with $\mu = \tau'^\circ \neq 0$. (Moreover in this case we have further that $\mu \neq 1$). (*Case* 2): Suppose that for every $b \neq 0, \neq 1$, $b \Delta b'^\circ \Delta b^{\circ\prime} = 0$. Then from D, for $a \neq 0, \neq 1$ we have $aa^{\circ\prime}a'^\circ = 1$, and the theorem is proved with $\mu = 1$.

T4′. (= *dual of* T4). $a \Delta a'^\circ \Delta a^{\circ\prime} = \tau = \text{constant} \neq 1, \ [a \neq 0, \neq 1]$.

It is seen from T4 and T4′ that the constants μ and τ are *dual* elements, and hence, when obtaining the dual of a field proposition involving them, each must be replaced by the other. It is further easily verified that these constants μ and τ turn out, in a field, to have the realizations "-1" and "$2(=1+1)$" respectively. (See footnote to T19′). In the proof of T4 we have further shown

T5. (i) $\mu = 1 \rightleftarrows \tau = 0$

(ii) *If* $\mu \neq 1$ (*i. e.*, $\tau \neq 0$), *then* $\mu^\circ = \tau'$.

T6. $\mu' = \mu$.

Proof. By T4 $\mu \neq 0$, therefore μ' exists and is given by

$$\mu' = (aa^{\circ\prime}a'^\circ)' \ [a \neq 0, \neq 1]$$
$$= a'(a^{\circ\prime})'(a'^\circ)', = \mu$$

by T's 3, 4.

T6′. $\tau^\circ = \tau$.

T7. $aa = 1 \rightarrow a = 1$ *or* $a = \mu$.

Conversely, $a = 1$ *or* $a = \mu : \rightarrow aa = 1$.

Proof. From T4, since $\times$ is a group, $a^{\circ\prime}a^{\prime\circ} \neq 1$ unless $a = \mu$. Hence $a^{\prime\circ} \neq a^\circ$ unless $a = \mu$, i. e., $aa \neq 1$ unless $a = \mu$ or $a = 1$ (T's 4, 3, 2). The converse: For $a = 1$ this follows by T2. For $a = \mu \neq 1$,

$$\mu = bb^{\circ\prime}b^{\prime\circ}$$
$$\mu = b^\prime b^{\prime\circ\prime}b^{\prime\prime\circ}$$

for $b \neq 0, \neq 1$ (T's 4, 3). Therefore $\mu\mu = bb^{\circ\prime}b^{\prime\circ}b^\prime b^{\prime\circ\prime}b^\circ = 1$ (T3).

T7′. $a \Delta a = 0 \rightarrow a = 0$ *or* $a = \tau$.

Conversely, $a = 0$ *or* $a = \tau : \rightarrow a \Delta a = 0$.

In proving T7 we have also established

T8. *If* $\mu = 1$, *then for all* $a \neq 0, \neq 1$, $a^{\circ\prime}a^{\prime\circ} \neq 1$. *If* $\mu \neq 1$, *then* $a^{\circ\prime}a^{\prime\circ} = 1$ *only for* $a = \mu$.

T8′. *If* $\tau = 0$, *then for all* $a \neq 0, \neq 1$, $a^{\prime\circ} \Delta a^{\circ\prime} \neq 0$. *If* $\tau \neq 0$, *then* $a^{\prime\circ} \Delta a^{\circ\prime} = 0$ *only for* $a = \tau$.

T9. $a^{\prime\circ\prime} = a^{\circ\prime\circ}$ $[a \neq 0, \neq 1]$.

Proof. $a^\prime a^{\prime\circ\prime}a^{\prime\prime\circ} = \mu$ ($a^\prime$ for a in T4), $= a^\prime a^{\prime\circ\prime}a^\circ$ (Σ3).
$a^\circ a^{\circ\prime}a^{\circ\prime\circ} = \mu$ (a° " a "), $= a^\circ a^\prime a^{\circ\prime\circ}$.

The theorem follows from these two expressions for μ, by T3. It is to be noted that T9 is a self-dual theorem.

We now define a^*, the (field) *complement* of a by

Definition E1.

$$a^* = \begin{cases} a^{\prime\circ\prime} & \text{if } a \neq 0, \neq 1 \\ 1 & \text{if } a = 0 \\ 0 & \text{if } a = 1. \end{cases}$$

T10.

$$a^* = \begin{cases} a^{\circ\prime\circ}\ (= a^{\prime\circ\prime}) & \text{if } a \neq 0, \neq 1 \\ 1 & \text{if } a = 0 \\ 0 & \text{if } a = 1. \end{cases}$$

Proof. T9, E1.

Comparing E1 and T10 it is seen that the operation * is self-dual and

hence * is left unchanged in obtaining the dual of any field proposition involving this operation.

T11. *a^* is a unique element of F for each a of F.*

Proof. B, B′, T10.

T12. $a^{**} = a$.

Proof. $0^{**} = 1^* = 0$; $1^{**} = 0^* = 1$; for $a \neq 0, \neq 1$, $a^{**} = a^{'\circ 1 1 \circ '} = a$ (T's 10, 3).

T13. $a^* = b^* \leftrightarrow a = b$

$a^* = b \leftrightarrow a = b^*$.

Proof. T's 11, 12.

T14. $a^{'\circ} = a^{*'}$ $[a \neq 0, \neq 1]$.

Proof. $a^{*'} = a^{'\circ 1 1} = a^{'\circ}$ (T's 10, 3).

T14′. $a^{\circ'} = a^{*\circ}$ $[a \neq 0, \neq 1]$.

T15. $a^{'\circ} = a^{\circ *}$ $[a \neq 0, \neq 1]$.

Proof. $a^{\circ *} = a^{\circ\circ 1\circ} = a^{'\circ}$ (T's 10, 3).

T15′. $a^{\circ'} = a^{'*}$ $[a \neq 0, \neq 1]$.

T16. $a^{\circ *} = a^{*'}$ $[a \neq 1]$.

Proof. $0^{\circ *} = 0^* = 1$; $0^{*'} = 1' = 1$ (T's 3, 10); for $a \neq 0, \neq 1$, $a^{\circ *} = a^{'\circ}, = a^{*'}$ (T's 10, 14, 3).

T16′. $a^{'*} = a^{*\circ}$ $[a \neq 0]$.

From Theorems 14-16′ and T's 4, 4′ one easily culls the following theorem, symmetric in the three unary operations ', °, * which we call the

T17 *Unary Operations Theorem.*[2] *Let $\sigma(=\sigma_1), \sigma_2, \sigma_3$ denote (in any desired order) the three unary operations ', °, * and let a, b be $\neq 0, \neq 1$. Then*

(i) $a^\sigma \neq 0, \neq 1$

(ii) $a = b \rightleftarrows a^\sigma = b^\sigma$

---

[2] In [1] it is shown that in the case of a general ring, R, Theorem 17 holds in $\mathfrak{U}$, the *double-unity* subset of R, i. e., the subset of those elements of R which are unity elements both with respect to the group $\times$ and with respect to its dual, Δ.

(iii) $a^{\sigma\sigma} = a$

(iv) $a^{\sigma_1\sigma_2\sigma_3} = a^{\sigma_2}$

(v) $a^{\sigma_1\sigma_2\sigma_1} = a^{\sigma_3}$

(vi) $a^{\sigma_1\sigma_2} = a^{\sigma_2\sigma_3}$

(vii) $a^{\sigma_1}a^{\sigma_2}a^{\sigma_3} = \mu$

(viii) $a^{\sigma_1} \Delta a^{\sigma_2} \Delta a^{\sigma_3} = \tau.$

By comparison with the definition E1, (v) above shows that for $a \neq 0, \neq 1$ any one of the three unary operations °, ', * is definable in terms of the other two, and moreover in each instance by equations of the same form, namely of the form given in E1.

T18. $\tau = \mu^*$
$\mu = \tau^*.$

Proof. The second formula follows from the first by T12. Consider then the first, $\tau = \mu^*$. If $\tau = 0$ then $\mu = 1$, and the theorem is immediate, (T's 5, 10). For $\tau \neq 0$ (i. e., $\mu \neq 1$, by T5), $\mu^\circ = \tau'$ (D, T's 4, 4′). Therefore $\tau = \tau^\circ = \mu^{\circ\prime\circ} = \mu^*$ (T's 6′, 3, 10).

T19. $\mu^{*\circ} = \mu^* = \tau.$

Proof. T's 18, 6′.

T19′.[3] $\tau^{*\prime} = \tau^* = \mu.$

T20. $ab = a \Delta a^\circ b^*$ $[a \neq 1].$

Proof. Putting $a^\circ b' = q'^\circ$, $b' = a^{\circ\prime}q'^\circ$ in C′, we get

(1) $a \Delta a^{\circ\prime\prime}q'^{\circ\prime} = aq'^{\circ\circ\prime}$ $[a \neq 0, \neq 1;\ a^{\circ\prime\prime}q'^{\circ\prime} \neq 0;\ q'^\circ \neq 1].$

Using T's 3, 10, (1) establishes

---

[3] $\mu = 1$ (i. e., $\tau = 0$) only for fields of characteristic 2. For fields of characteristic $\neq 2$ the *center* (element), ω, plays a more symmetrical role than either μ or τ, where *Definition.*

$$\omega = \begin{cases} 1 \text{ for a field of characteristic 2.} \\ \mu^\circ [= \tau' = (aa^{\circ\prime}a'^\circ)^\circ = (a \Delta a'^\circ \Delta a^{\circ\prime})'] \\ \text{for a field of characteristic} \neq 2. \text{ (Compare with postulate D).} \end{cases}$$

One then easily has the

THEOREM. *For fields of characteristic* $\neq 2$ $\omega^\circ = \mu$, $\omega' = \tau$, $\omega^* = \omega$.

The center of a field of characteristic p is readily computed to be given by

$$\omega \begin{cases} = \frac{1}{2}(= (1+1)') \text{ if } p = 0 \\ = (p-1)/2 \text{ (i.e., } \underbrace{1 + 1 + \cdots + 1}_{(p-1)/2 \text{ terms}}) \text{ if } p \neq 0. \end{cases}$$

(2) $\quad a \Delta a^{\circ} q^{*} = aq \quad (\text{for } a \neq 0, \neq 1;\ q \neq 0, \neq 1).$

The domain of (2), however, unlike that of (1), also permits $a = 0$, $q = 0$, or $q = 1$. For each of these cases (2) is immediately true by T's 3, 2 and E1, which establishes T20.

T20′. $\quad a \Delta b = a(a' \Delta b^{*}) \qquad [a \neq 0].$

T21. $\quad (a \Delta b)^{*} = a^{*} b^{*}.$

Proof. By T20 and C we have respectively

$$ab = a \Delta a^{\circ} b^{*} \qquad [a \neq 1].$$

$$ab = a \Delta (a' \Delta b^{\circ})^{\prime\circ} \qquad [a \neq 0, \neq 1;\ b \neq 1;\ a' \Delta b^{\circ} \neq 0].$$

Since ab is defined for all a, b of F, we then have, by T3,

$$a^{\circ} b^{*} = (a' \Delta b^{\circ})^{\prime\circ} \qquad [a \neq 0, \neq 1;\ a' \Delta b^{\circ} \neq 0]$$

$$= (a^{\prime\circ} \Delta b)^{\circ\prime\circ} \qquad [\quad \text{“} \quad \text{“} \quad \text{“} \quad].$$

Putting $a^{\prime\circ} = q$ and using T10 this becomes

(1) $\quad q^{*} b^{*} = (q \Delta b)^{*} \quad (\text{for } q \neq 0, \neq 1;\ b \neq 1;\ q \Delta b \neq 0).$

The domain of (1), however, permits all values of q and b. For each of the cases $q = 0$, $q = 1$, or $b = 1$, (1) is directly verified by T's 2, 10. And for the case $q \Delta b = 0$ (1) is also true, since

$$q \Delta b = 0 (q \neq 0, \neq 1;\ b \neq 0, \neq 1) \begin{cases} \rightarrow (q \Delta b)^{*} = 1 \quad (\text{E1}). \\ \rightarrow q = b^{\circ}, \rightarrow q^{\prime\circ} = b^{\circ\prime\circ}, \rightarrow q^{\prime\circ\prime} b^{\circ\prime\circ} \\ \qquad = 1, \rightarrow q^{*} b^{*} = 1 \quad (\text{T's } 3, 10). \end{cases}$$

This proves T21.

T21′. $\quad (ab)^{*} = a^{*} \Delta b^{*}.$

T22. $\quad (a^{\circ} b^{\circ})^{\circ} = (a' \Delta b')' \quad [a \neq 0, \neq 1;\ b \neq 0, \neq 1;\ a' \Delta b' \neq 0].$

Proof. By T's 21, 12

$$(a^{\circ} b^{\circ})^{*} = a^{\circ *} \Delta b^{\circ *} \qquad [a \neq 1, b \neq 1].$$

The theorem then follows at once from T's 10, 2, 3. Theorem 22, it is observed, is another self-dual theorem.

T23. $\quad (a^{\circ} b^{\circ})^{\circ} = a(a^{\prime\circ} b)^{\circ} \qquad [a \neq 0, \neq 1;\ a^{\circ} b^{\circ} \neq 1].$

Proof. For $b = 0$ (and $a \neq 0, \neq 1$) the theorem follows from T's 2, 3. Therefore, comparing the domains of T's 23 and 22, we have only to prove that

$$(1) \quad a(a'^{\circ}b)^{\circ} = (a^{r} \Delta b')' \quad \text{for} \quad (a \neq 0, \neq 1;\ b \neq 0, \neq 1;\ a^{\circ}b^{\circ} \neq 1).$$

Under these restrictions, however, C′ gives

$$(2) \qquad (a' \Delta b')' = a''(a'^{\circ}b'')^{\circ''},$$

which yields (1) and hence T23, by use of T3.

T23′. $(a' \Delta b')' = a \Delta (a^{\circ'} \Delta b)' \qquad [a \neq 0, \neq 1;\ a' \Delta b' \neq 0].$

4. Definition of $+$. Of several available definitions for field addition, $+$, we have chosen one which is commutative by inspection. Considerably more compact (but unsymmetric) formulas for $+$ are derived as theorems. The "singularities" of different such formulas for addition will in general be different.

Definition E2.

$$\begin{cases} a + b = a \Delta b \Delta a^{\circ}b^{\circ} & [a \neq 1, b \neq 1]. \\ a + 1 = 1 + a = a \Delta a'^{\circ} & [a \neq 0, \neq 1]. \\ 1 + 1 = \tau \\ 1 + 0 = 0 + 1 = 1. \end{cases}$$

We note (see **1**) that the dual of the definition E2 furnishes a definition of $a \dot{+} b$.

Definition E2′.

$$\begin{cases} a \dot{+} b = ab(a' \Delta b') & [a \neq 0, b \neq 0]. \\ a \dot{+} 0 = 0 + a = aa^{\circ'} & [a \neq 0, \neq 1]. \\ 0 \dot{+} 0 = \mu. \\ 0 \dot{+} 1 = 1 + 0 = 0. \end{cases}$$

It is quite immaterial whether we elect to show that $(F, +, \times)$, with the $+$ from E2, or that $(F, \dot{+}, \Delta)$, with $\dot{+}$ from E2′, is equivalent to the usual definition of field, since either follows from the other by the duality theorem (i. e., from the dual-pair nature of our postulates). We shall prove the former.

Theorems 24 to 27 are immediate consequences of E2, B, B′.

T24. *$a + b$ is a unique element of F for each a, b of F.*

T25. $a + b = b + a$.

T26. $a + b = 1,\ a \neq 1,\ b \neq 1 : \rightarrow a^\circ b^\circ = 1$.

Conversely, $a^\circ b^\circ = 1,\ a \neq 1,\ b \neq 1 : \rightarrow a + b = 1$.

T27. $a + b = 0,\ a \neq 1,\ b \neq 1 : \rightarrow a^\circ \Delta b^\circ = a^\circ b^\circ$.

Conversely, $a^\circ \Delta b^\circ = a^\circ b^\circ,\ a \neq 1,\ b \neq 1 : \rightarrow a + b = 0$.

T28. $a + b = a \Delta a'^\circ b \qquad [a \neq 0, \neq 1]$.

Proof. For $b = 1$ the theorem is true by E2, T2. For $a \neq 0, \neq 1$

$$a \Delta a'^\circ b = a \Delta b \Delta b^\circ a'^{\circ *} = a \Delta b \Delta b^\circ a^\circ, \tag{1}$$

by T's 20, 10, 2. The right of (1) is $a + b$ for $a \neq 0, \neq 1, b \neq 1$, which proves the theorem.

T29. $a + b = a(\mu a'b)^* \qquad [a \neq 0]$.

Proof. *Case* 1, $a = 1$. Here, by T2, we must show that

$$1 + b = (\mu b)^*. \tag{1}$$

For $b = 0$ or for $b = 1$, (1) follows from T's 2, 10, 18, E2. For $b \neq 0, \neq 1$ write $\mu = b'b'^{\circ}b'^{\prime\circ}$ (by T's 4, 3), whence, using T10, (1) becomes

$$1 + b = (b^* b'^{\prime\circ})^*$$

or, by T's 21′, 12, 10,

$$1 + b = b \Delta b'^\circ. \tag{2}$$

But (2) is true by E2, which proves case 1. There remains only to prove the theorem for $a \neq 0, \neq 1$. Putting $\mu = aa'^\circ a^{\circ\prime}$ and using T's 28, 10, 3 we have to show that

$$a \Delta a'^\circ b = a(a'^\circ a^{\circ\prime} b)^*. \tag{3}$$

If $a'^\circ a^{\circ\prime} b = 1$ each side of (3) reduces to 0 (by T's 3, 10, 2). If $a'^\circ a^{\circ\prime} b \neq 1$ we may use C′ on the left of (3) and hence have to show that

$$a(a^\circ a'^{\circ\prime} b')^{\circ\prime} = a(a'^\circ a^{\circ\prime} b)^*. \tag{4}$$

But (4) follows at once from T10, since the left side may be written $a(a^{\circ\prime} a'^\circ b)^{\prime\circ\prime}$. This proves T29.

T30. $a^*(ba^{*\prime})^* = b^*(ab^{*\prime})^* \qquad [a \neq 1, b \neq 1]$.

Proof. From C′ and the commutativity of Δ we have

$$a(a^{\circ}b')^{\circ'} = b(b^{\circ}a')^{\circ'} \qquad [a \neq 0, \neq 1;\ b \neq 0, \neq 1;\ a^{\circ}b' \neq 1]$$

i. e., $\quad b'(a^{\circ'}b)^{\circ'^{\circ'}} = a'(b^{\circ'}a)^{\circ'^{\circ'}}$ [" " "]

i. e., (1) $\quad b'(a^{\circ'}b)^{*} = a'(b^{\circ'}a)^{*}$ for [" " "].

The domain of (1) also permits $a^{\circ}b' = 1$ ($\rightleftarrows b^{\circ}a' = 1$), in which case (1) is true (each side reducing to 0) by T's 3, 10, 2. Hence we have proved the formula (1) for its domain; that is, putting $b' = r^{*}$, $a' = s^{*}$, we have proved

(2) $\qquad r^{*}(sr^{*'})^{*} = s^{*}(rs^{*'})^{*}$ for $\quad (r \neq 0, \neq 1;\ s \neq 0, \neq 1).$

The domain of (2) also permits $r = 0$ as well as $s = 0$. In each of these cases, however, (2) is at once true by T's 10, 2, 3, (each side reducing to s^{*}, respectively to r^{*}). Hence (2) is proved for its domain, which completes T30.

T31. $\quad (a + b) + c = a + (b + c).$

Proof. If any of the elements $a, b, c, = 0$ the theorem is true by E2. Hence we assume $a, b, c, \neq 0$. *Case* 1: $a + b \neq 0$, $b + c \neq 0$. We may here use T29 for all $+$'s, and have

$$(a + b) + c = b(\mu b'a)^{*}\{\mu b'(\mu b'a)^{*'}c\}^{*}$$

$$a + (b + c) = b(\mu b'c)^{*}\{\mu b'(\mu b'c)^{*'}a\}^{*}.$$

Hence we must establish that

(1) $\qquad b(\mu b'a)^{*}\{\mu b'(\mu b'a)^{*'}c\}^{*} = b(\mu b'c)^{*}\{\mu b'(\mu b'c)^{*'}a\}^{*}.$

Put $\mu b'a = r$, $\mu b'c = s$. Now $r \neq 1$, $s \neq 1$ since $r = 1 \rightarrow a + b = 0$ (T's 29, 10, 3), and similarly $s = 1 \rightarrow b + c = 0$, contrary to hypothesis. Therefore (1) is proved since, using T3, it is equivalent to

$$r^{*}(sr^{*'})^{*} = s^{*}(rs^{*'})^{*}$$

which is true by T30. This proves case 1. *Case* 2: $a + b = 0$, $b + c \neq 0$. (This is of course equivalent to the case: $a + b \neq 0$, $b + c = 0$). Now $a + b = 0 \rightarrow \mu ab' = 1$ (T's 29, 10). Therefore we must show that

$$0 + c = b(\mu b'c)^{*}\{\mu b'(\mu b'c)^{*'}a\}^{*} \quad \text{if} \quad \mu ab' = 1,$$

i. e. (by T's 6, 3, 2, E2) that

(2) $$c = b(\mu b'c)^*(\mu b'c)^{*\prime *}.$$

Putting $b'c = h$, noticing that $\mu h \neq 0, \neq 1$ (by T's 29, 3, since $b + c \neq 0$), and using T's 3, 10, 17, (2) reduces to

$$h' = (\mu h)'^{\circ}(\mu h)^{\circ\prime}.$$

This however is evident, since by T4

$$\mu = \mu h(\mu h)'^{\circ}(\mu h)^{\circ\prime}.$$

This proves case 2. *Case* 3: $a + b = 0$, $b + c = 0$. Here, using T's 29, 10, we must prove that

$$\mu a'b = \mu c'b (= 1) \ : \rightarrow c = a.$$

This is immediate by T3, and completes the proof of T31.

T32. $0 + a \ (= a + 0) = a$.

Proof. E2, T's 2, 3.

T33. *$a + x = c$ has a solution.*

Proof. For $a = 0$, $x = c$ is a solution (T32). For $a \neq 0$,

$$x = a\mu(ca')^*$$

is a solution, by T's 29, 3, 6, 12.

T34. $a(b + c) = ab + ac$.

Proof. If a, b or $c = 0$ the theorem follows from T2 and T32. Assume, then, $a \neq 0$, $b \neq 0$, $c \neq 0$, which further implies $ab \neq 0$, $ac \neq 0$ (T3). Then by T29,

(1) $$a(b + c) = ab(\mu b'c)^*$$

and

(2) $$ab + ac = ab\{\mu(ab)'ac\}^*, = ab(\mu b'c)^*.$$

The theorem follows by comparing (1) and (2).

From B and T's 2, 24, 25, 31, 32, 33, 34 one then has

T35. *With $+$ defined by E2, $(F, +, \times)$ is a field with* 1 *as its unit and* 0 *as its zero element.*

5. Equivalence. Theorem 35 together with the observation immediately following the postulates establishes the equivalence, in the sense of inter-

definability, of the classical $(F, +, \times; 0, 1)$ and the dual-symmetric $(F, \Delta, \times; 0, 1)$ characterization of the concept field via the connecting definitions

$$(F, +, \times; 0, 1) \parallel (F, \Delta, \times; 0, 1)$$

$F \leftarrow$	defined by	$\rightarrow F$	
$0 \leftarrow$	"	$\rightarrow 0$	
$1 \leftarrow$	"	$\rightarrow 1$	
$a \times b \leftarrow$	"	$\rightarrow a \times b$	
$a + b - ab \leftarrow$	"	$\leftarrow a \Delta b$	
$a + b \rightarrow$	"	$\rightarrow a \Delta b \Delta a^{\circ} b^{\circ}$	(for $a \neq 1, b \neq 1$)
$a + 1, = 1 + a \rightarrow$	"	$\rightarrow a \Delta a'^{\circ}$	(" $a \neq 0, \neq 1$)
$1 + 1 \rightarrow$	"	$\rightarrow h \Delta h'^{\circ} \Delta h^{\circ'}$	(" $h \neq 0, \neq 1$)
$1 + 0, = 0 + 1 \rightarrow$	"	$\rightarrow 1$.	

But more than this is true,—the two characterizations are *strongly*[4] equivalent, that is, also the (above) *definitions* used in each system (to establish the equivalence) are provable as theorems of the other system. Only two such theorems come into question here, namely

THEOREM EQ 1. $a + b - ab = a \Delta b$,

a theorem of our dual-symmetric characterization $(F, \Delta, \times)$, and

THEOREM EQ 2.

$$\begin{aligned} a + b &= a \Delta b \Delta a^{\circ} b^{\circ} && (\textit{for } a \neq 1, b \neq 1) \\ a + 1 = 1 + a &= a \Delta a'^{\circ} && (\text{ " } a \neq 0, \neq 1) \\ 1 + 1 &= h \Delta h'^{\circ} \Delta h^{\circ'} && (\text{ " } h \neq 0, \neq 1) \\ 0 + 1 = 1 + 0 &= 1, \end{aligned}$$

a theorem of the classical $(F, +, \times)$ characterization of field. The latter is immediately verified upon substituting $a + b - ab$, $a/a - 1$ and $1/a$ for $a \Delta b$, a° and a' respectively. We prove Theorem Eq 1, or, what is clearly the same, the formula

$$(1) \qquad a + b = (a \Delta b) + ab.$$

Proof. For each of the cases $a = 0$, $a = 1$, $b = 0$, $b = 1$, (1) is true by E2, T2. Assume, then, $a \neq 0, \neq 1$; $b \neq 0, \neq 1$. *Case* 1: $a \Delta b \neq 0$. In this case, by T28,

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[4] The explicit recognition of the important distinction between mere "interdefined" and "strong" equivalence of postulates (and related observations on equivalence) is due to A. Tarski in a paper not yet published.

(2) $$(a \Delta b) + ab = a \Delta b \Delta (a \Delta b)'^{\circ} ab$$

(3) $$a + b = a \Delta b \Delta a^{\circ} b^{\circ}.$$

Hence, simplifying by T3, we must prove that

$$(a \Delta b)'^{\circ} ab = a^{\circ} b^{\circ}$$

or, using C′, that

(4) $$\{a'(a^{\circ} b')^{\circ}\}^{\circ} ab = a^{\circ} b^{\circ}.$$

Now $a'(a^{\circ}b')^{\circ} \neq 1$, since by T3

$$a'(a^{\circ}b')^{\circ} = 1 \rightarrow a^{\circ}b' = a^{\circ}, \rightarrow b = 1,$$

contrary to hypothesis. Therefore we may use T23, and (4) becomes

$$a^{\circ} b' \{(a^{\circ'} b)^{\circ} a'^{\circ}\}^{\circ} ab = a^{\circ} b^{\circ}$$

i. e., by T3,

(5) $$a\{(a^{\circ'} b)^{\circ} a'^{\circ}\}^{\circ} = b^{\circ}.$$

Again $(a^{\circ'}b)^{\circ}a'^{\circ} \neq 1$, since

$$(a^{\circ'}b)^{\circ}a'^{\circ} = 1 \rightarrow a^{\circ'}b = a'^{\circ'^{\circ}} = a^{\circ'} \text{ (by T's 9, 3)}, \rightarrow b = 1,$$

contrary to hypothesis. So we may again apply T23, and (5) becomes

(6) $$aa'(a^{\circ}a^{\circ'}b)^{\circ} = b^{\circ},$$

which is obviously true, by T3. This proves case 1. *Case* 2: $a \Delta b = 0$, (equivalent to $b = a^{\circ}$). Using E2 and T32 we must here show that

(7) $$aa^{\circ} = a \Delta a^{\circ} \Delta aa^{\circ}.$$

But (7) is obvious by T3. This completes Theorem Eq 1, and establishes the strong-equivalence of the ordinary $(F, +, \times)$ and the dual-symmetric $(F, \Delta, \times)$ characterizations of the concept field.

6. Set II of postulates. The primitive notions of set II are $(F, \Delta, \times)$, where $\Delta, \times$ as before, correspond (in realization) to the two dual field multiplications. The field complement, *, as a definition, is here involved in the formulation of certain of the postulates, and this requires that the dual of a proposition

$$P(1, 0; \times, \Delta, {}^{\circ}, {}', {}^*)$$

be *defined* as

$$P(0, 1; \Delta, \times, {}', {}^{\circ}, {}^*),$$

where, it is to be noted, the invariance of * (unlike set I) is here a part of the *definition* of *dl P*. The postulates of set II are then seen to occur in dual pairs, (with N and P7 self-dual), and hence, as in I, it is only necessary to prove one of a pair of dual theorems. References are given in full with the exception of those to N, P1, P1′, which are omitted. The postulates are:

N. *F contains at least three elements.*

P1. *ab is a unique element of F whenever a, b are in F.*

P1′. *$a \Delta b$ is a unique element of F whenever a, b are in F.*

P2. $a(bc) = (ab)c$.

P2′. $a \Delta (b \Delta c) = (a \Delta b) \Delta c$.

P3. $ab = ba$.

P3′. $a \Delta b = b \Delta a$.

P4. *There is an element* 1 *in F such that* $a1 = a$ *for every a in F.*

P4′. *There is an element* 0 *in F such that* $a \Delta 0 = a$ *for every a in F.*

P5. *For each* $a \neq 0$ *in F there is an* a' *in F such that* $aa' = 1$.

P5′. *For each* $a \neq 1$ *in F there is an* a° *in F such that* $a \Delta a^\circ = 0$.

Definition D1.

$$\begin{cases} a^* = a^{\circ\prime\circ} & [a \neq 0, \neq 1]. \\ 1^* = 0. \\ 0^* = 1. \end{cases}$$

P6. $ab = a \Delta a^\circ b^*$ $\quad [a \neq 1]$.

P6′. $a \Delta b = a(a' \Delta b^*)$ $\quad [a \neq 0]$.

P7. $(a^\circ a' a^*)^\circ = (b' \Delta b^\circ \Delta b^*)'$ $\quad [a \neq 0, \neq 1;\ b \neq 0, \neq 1]$.

T0. (THEOREM 0). $0 \neq 1$.

Proof. (1) $a0 = 0$, since: if $a \neq 1$, then

$$a0 = a \Delta a^\circ 0^* = a \Delta a^\circ 1 = a \Delta a^\circ = 0,$$

by P6, D1, P4, P5′: and if $a = 1$, then $01 = 0$, by P4, P3. If $0 = 1$, we have the contradiction: for $a \neq 0$, $a = a1 = a0 = 0$, by N. P4, hypothesis, and (1).

T1. $(F, \times)$ *without* 0 *is an Abelian group with* 1 *as its unit element.*

Proof. T0, P's 1, 2, 3, 4, 5.

T1′. (F, Δ) *without* 1 *is an Abelian group with* 0 *as its unit element.*

T2. $0a = a0 = 0$.

Proof. P3 and (1) of T0.

T2′. $1 \Delta a = a \Delta 1 = 1$.

T3. $a^{**} = a$.

Proof. If $a \neq 0, \neq 1$, then $a^{**} = a^{\circ 1 \circ 0 1 \circ} = a^{\circ 11 \circ} = a^{\circ\circ} = a$, by D1, T's 1′, 1, 1′. If $a = 0$, then $0^{**} = 1^{*} = 0$, by D's 1, 1. If $a = 1$, then $1^{**} = 0^{*} = 1$, by D's 1, 1.

T4. $(ab)^{*} = a^{*} \Delta b^{*}$.

Proof. If $a \neq 0, \neq 1$, then $a^{\circ} b^{*} = a^{\circ} \Delta ab$; by P6, T1′; hence

$$b^{*} = a^{\circ 1}(a^{\circ} \Delta ab) = a^{\circ 1}(a^{\circ 11} \Delta ab) = a^{\circ 1}\{a^{\circ 11} \Delta (ab)^{**}\} = a^{\circ 1} \Delta (ab)^{*}$$

by T's 1, 1, 3, P6′; hence, $(ab)^{*} = a^{\circ 1 \circ} \Delta b^{*} = a^{*} \Delta b^{*}$, by T1′, D1. If $a = 1$, then $(1b)^{*} = b^{*} = 0 \Delta b^{*} = 1^{*} \Delta b^{*}$, by T's 1, 1′, D1. If $a = 0$, then $(0b)^{*} = 0^{*} = 1 = 1 \Delta b^{*} = 0^{*} \Delta b^{*}$, by T2, D1, T2′, D1.

T4′. $(a \Delta b)^{*} = a^{*} b^{*}$.

T5. $a^{\circ} a'^{*} = 1 \qquad [a \neq 0, \neq 1]$.

Proof. $1 = aa' = a \Delta a^{\circ} a'^{*}$, by T1, P6; hence, $a^{\circ} a'^{*} = 1 \Delta a^{\circ} = 1$, by T's 1′, 2′.

T5′. $a' \Delta a^{\circ *} = 0, \qquad [a \neq 0, \neq 1]$.

T6. $a'^{*} = a^{\circ 1} \qquad [a \neq 0, \neq 1]$.

Proof. T's 5, 1.

T6′. $a^{\circ *} = a'^{\circ} \qquad [a \neq 0, \neq 1]$.

T7. $a^{*} = a'^{\circ 1} = a^{\circ 1 \circ} \qquad [a \neq 0, \neq 1]$.

Proof. $a^{\circ 1 \circ} = a^{*} = a''^{*} = a'^{\circ 1}$, by D1, T's 1, 6.

T8. $a(a'b^{*})^{*} = b(b'a^{*})^{*} \qquad [a \neq 0, b \neq 0]$.

Proof.

$$\begin{aligned} a(a'b^*)^* &= a(a'^* \Delta b^{**}) = a(a'^* \Delta b) = a(b \Delta a'^*) \\ &= a\{b(b' \Delta a'^{**})\} = a\{b(b' \Delta a')\} = b\{a(b' \Delta a')\} \\ &= b\{a(a' \Delta b')\} = b\{a(a' \Delta b'^{**})\} = b(a \Delta b'^*) \\ &= b(a^{**} \Delta b'^*) = b(a^*b')^* = b(b'a^*)^*, \end{aligned}$$

by T's 4, 3, 1′, P6′, T's 3, 1, 1′, 3, P6′, T's 3, 4, 1.

T8′. $a \Delta (a^\circ \Delta b^*)^* = b \Delta (b^\circ \Delta a^*)^* \qquad [a \neq 1, b \neq 1]$.

T9. $a^\circ a' a^* = a$ *constant* $\qquad [a \neq 0, \neq 1]$.

Proof. P7, T's 1, 1′.

T9′. $a' \Delta a^\circ \Delta a^* = a$ *constant* $\qquad [a \neq 0, \neq 1]$.

T10. *If* $a \neq 0, \neq 1$ *then:* $a^\circ \neq 0, \neq 1$; $a' \neq 0, \neq 1$; $a^* \neq 0, \neq 1$.

Proof. T's 1′, 1′, 1, 1′, 3, D1.

Definition D2. $\mu = a^\circ a' a^* \qquad [a \neq 0, \neq 1]$.

Definition D2′. $\tau = a' \Delta a^\circ \Delta a^* \qquad [a \neq 0, \neq 1]$.

T11. $\mu \neq 0$.

Proof. D2, T's 10, 1.

T11′. $\tau \neq 1$.

T12. (i) $\mu = a^\circ a' a^* = a a^{\circ\prime} a'^\circ$; $\tau = a' \Delta a^\circ \Delta a^* = a \Delta a'^\circ \Delta a^{\circ\prime}$ $[a \neq 0, \neq 1]$.

(ii) $\mu' = \mu$; $\tau^\circ = \tau$.

(iii) $\mu^\circ = \tau' \ [\mu \neq 1, \ (\rightleftarrows \tau \neq 0)]$.

(iv) $\mu^* = \tau$; $\tau^* = \mu$.

(v) $\mu\mu = 1$; $\tau \Delta \tau = 0$.

(vi) $\mu a = a^* a^\circ \ [a \neq 1]$; $\tau \Delta a = a^* \Delta a' \ [a \neq 0]$.

(vii) $\mu a' = a'^* a^{\circ *} \ [a \neq 0, \neq 1]$; $\tau \Delta a^\circ = a^{\circ *} \Delta a'^*$ $[a \neq 0, \neq 1]$.

Proof. Only the first parts require proof, since the second parts are their duals. (i): by D2 (a° for a) and T6′. (ii): by i, T's 1, 7. (iii): by P6, D2, D2′. (iv): by D1, iii, T1, ii. (v) by ii, T1. (vi): by i, T1. (vii): by vi, T6′.

T13. $a(\mu a')^* = (\mu a)^* \qquad [a \neq 0]$.

Proof.

$$a(\mu a')^* = a(a'^* a^{\circ *})^* = a(a'^{**} \Delta a^{\circ **}) = a(a' \Delta a^{\circ **})$$
$$= a \Delta a^{\circ *} = a^{**} \Delta a^{\circ *} = (a^* a^\circ)^* = (\mu a)^*,$$

by T12 (vii), T's 4, 3, P6′, T's 3, 4, 12 (vi).

T13′. $a \Delta (\tau \Delta a^\circ)^* = (\tau \Delta a)^* \qquad [a \neq 1]$.

T14. $a\{\mu a'(\mu b)^*\}^* = b\{\mu b'(\mu a)^*\}^* \qquad [a \neq 0, b \neq 0]$.

Proof. $(\mu a)\{(\mu a)'(\mu b)^*\}^* = (\mu b)\{(\mu b)'(\mu a)^*\}^*$, by T's 8, 11.

Hence

$$\mu a\{\mu' a'(\mu b)^*\}^* = \mu b\{\mu' b'(\mu a)^*\}^*,$$

by T1; hence the theorem, by T's 1, 11, 12 (ii), 1.

T14′. $a \Delta \{\tau \Delta a^\circ \Delta (\tau \Delta b)^*\}^* = b \Delta \{\tau \Delta b^\circ \Delta (\tau \Delta a)^*\}^* \; [a \neq 1, b \neq 1]$.

Definition D3. $\begin{cases} a + b = a(\mu a' b)^* & [a \neq 0]. \\ 0 + a = a. \end{cases}$

T15. $a + 1 = 1 + a$.

Proof. If $a \neq 0$, then

$$a + 1 = a(\mu a' 1)^* = a(\mu a')^* = (\mu a)^* = 1(\mu a' a)^* = 1 + a,$$

by D3, T's 1, 13, 1, D3. If $a = 0$, then

$$1 + 0 = 1(\mu 1' 0)^* = (\mu 0)^* = 0^* = 1 = 0 + 1,$$

by D3, T's 1, 2, D1, D3.

T16. *$a + b$ is a unique element of F for all a, b of F.*

Proof. D3, T1.

T17. $a + 0 = 0 + a = a$.

Proof. If $a \neq 0$, then

$$a + 0 = a(\mu a' 0)^* = a 0^* = a 1 = a = 0 + a,$$

by D3, T2, D1, T1, D3. If $a = 0$, then $0 + 0 = 0$, by D3.

T18. $a(b + c) = ab + ac$.

Proof. If $a \neq 0$, $b \neq 0$, then

$$ab + ac = ab\{\mu(ab)' ac\}^* = ab(\mu b' c)^* = a\{b(\mu b' c)^*\} = a(b + c),$$

by D3, T's 1, 1, D3. If $a = 0$, $b \neq 0$, then

$$0(b + c) = 0 = 0 + 0 = 0b + 0c,$$

by T2, D3, T2. If $a \neq 0$, $b = 0$, then

$$a(0 + c) = ac = 0 + ac = a0 + ac,$$

by D's 3, 3, T2. If $a = 0$, $b = 0$, then

$$0(0 + 0) = 0 = 0 + 0 = 00 + 00,$$

by T2, D3, T2.

T19. $a + b = b + a$.

Proof. If $a \neq 0$, then

$$a + b = a1 + aa'b = a(1 + a'b) = a(a'b + 1) = aa'b + a1 = b + a,$$

by T's 1, 18, 15, 18, 1. If $a = 0$, the theorem is immediate, by T17.

T20. $a + (1 + b) = b + (1 + a)$.

Proof. If $a \neq 0$, $b \neq 0$, then

$$\begin{aligned} a + (1 + b) &= a + 1(\mu 1'b)^* = a + (\mu b)^* = a\{\mu a'(\mu b)^*\}^* \\ &= b\{\mu b'(\mu a)^*\}^* = b\{\mu b'1(\mu 1'a)^*\}^* = b + (1 + a), \end{aligned}$$

by D3, T1, D3, T's 14, 1, D3. If $a \neq 0$, $b = 0$, then

$$a + (1 + 0) = a + 1 = 1 + a = 0 + (1 + a),$$

by T's 17, 15, 17. If $a = 0$, $b \neq 0$, then

$$0 + (1 + b) = 1 + b = b + 1 = b + (1 + 0),$$

by T's 17, 15, 17. If $a = 0$, $b = 0$, then

$$0 + (1 + 0) = 0 + (1 + 0).$$

T21. $a + (b + c) = c + (b + a)$.

Proof. If $b \neq 0$, then

$$\begin{aligned} a + (b + c) &= bab' + (b + bb'c) = b\{ab' + (1 + b'c)\} \\ &= b\{b'c + (1 + ab')\} = bb'c + (b + bab') = c + (b + a), \end{aligned}$$

by T's 1, 18, 20, 18, 1. If $b = 0$, then

$$a + (0 + c) = a + c = c + a = c + (0 + a),$$

by T's 17, 19, 17.

T22. $a + (b + c) = (a + b) + c$.

Proof. T's 21, 19.

T23. *For each a there is an $\bar{a}$ such that $a + \bar{a} = 0$.*

Proof. If $a \neq 0$, then $\bar{a} = \mu a$. For,

$$a + \mu a = a\{\mu a'(\mu a)\}^* = a(\mu\mu)^* = a1^* = a0 = 0,$$

by D3, T's 1, 12(v), D1, T2. If $a = 0$, then $\bar{a} = 0$, by T17.

From T's 16, 17, 18, 19, 22, 23, N, P's 1, 2, 3, 4, 5 we have

T24. *With $+$ defined by D3, $(F, +, \times)$ is a field with 1 as its unit, and 0 as its zero element.*

With the help of P6 and T's 3, 4, 7, Postulate C of set I can readily be derived, and the equivalence of sets I and II established.

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A MATHEMATICAL THEORY OF DILATANCY.*

By M. Reiner.

Dilatancy is "that property of a material by which it increases in volume when it is stirred" or, more specifically, when it is sheared. Thus "in walking over wet sands the gloss disappears and the sand appears to dry out wherever the foot is placed." This is due to the increase in volume of the bulk of the sand, i. e., the increase of the *volume of the voids* into which the water is drawn. Dilatancy was first scientifically described by Reynolds in 1885, who coined the term. In the following a mathematical theory of dilatancy is expounded from the phenomenological point of view, i. e., no account will be taken of the internal structure of the material to which the phenomenon of dilatancy may be due. It will be shown that on purely mathematical grounds dilatancy should be present in a generalized Newtonian liquid.

1. A *Newtonian liquid* is a material which is perfectly elastic under a hydrostatic pressure (or tension) and perfectly viscous in distortion. By the latter is meant that all strainwork in distortion is entirely dissipated. This distinguishes the Newtonian liquid from that of Maxwell which has (relaxing) elasticity of shape. In a "*simple*" Newtonian liquid "*the resistance which arises from the lack of slipperiness of the parts of the liquid, other things being equal, is proportional to the velocity with which the parts of the liquid are separated from one another*" (Newton, *Principia*, Lib. ii, Sect. ix). In the "*generalized*" Newtonian liquid we drop the linearity and substitute for it any functional relationship between the deviators of stress and velocity-strain.[1] The generalized Newtonian liquid is accordingly one case of "non-linear" rheological bodies, the other important one being the non-linear or generalized Hookean, as we may say, elastic solid. The theory of the latter was treated by Hamel and brought to a conclusion by Murnaghan. It is bound up with the theory of finite strain. We define the strain tensor by $\epsilon_{xx} = \partial u/\partial x - \frac{1}{2}\{(\partial u/\partial x)^2 + (\partial v/\partial y)^2 + (\partial w/\partial z)^2\}$ etc. (i. e., the usual Cauchy definition, where u, v, w[2] are the components of the displacement vector) which, for small relative displacements, become $\partial u/\partial x$ etc.; and we define the strain of

* Received August 6, 1944.

[1] By "velocity-strain" we understand the space derivative of the velocity-vector, in contradistinction to the ordinary or *displacement*-strain.

[2] We shall use in the present paper Murnaghan's notation.

the velocity vector $(\dot{u}, \dot{v}, \dot{w})$ by $e_{xx} = \partial\dot{u}/\partial x$ etc., the higher terms being absent because the displacements $\dot{u}dt$ etc. in this case are infinitesimal; then evidently e_{xx} is generally not equal to $\dot{\epsilon}_{xx}$. This is expressed by Murnaghan in the words "the space derivative of the virtual displacement vector . . . is only in the infinitesimal theory (equal to) . . . the variation of the strain tensor." Hencky has shown how this difficulty can be overcome in the case of fixed principal axes of strain by a different definition of the strain, which for infinitesimal strain approximates the Cauchy strain. However, fortunately, we need not enter into these difficulties here. *The relative displacements upon which the stresses in a viscous liquid* (apart from the hydrostatic pressure) *depend are always infinitesimal by definition and* the theory is thereby considerably simplified.

2. The problem stated. In order to take account of the fundamentally different properties of a Newtonian liquid with respect to isotropic and deviatoric stress and strain components, we resolve the tensors of stress $T^*{}_s{}^r$, strain $\epsilon_s{}^r$ and velocity-strain $e_s{}^r$ as follows:

$$T^*{}_s{}^r = T^*\delta_s{}^r + T'{}_s{}^r, \tag{1}$$

$$\epsilon_s{}^r = \epsilon\delta_s{}^r + \epsilon'{}_s{}^r, \tag{2}$$

$$e_s{}^r = e\delta_s{}^r + e'{}_s{}^r. \tag{3}$$

This resolution is defined by

$$T'{}_a{}^a = \epsilon'{}_a{}^a = e'{}_a{}^a \equiv 0, \tag{4}$$

while the scalars $3T^*$, 3ϵ and $3e$ are the first invariant of the tensors $T^*{}_s{}^r$, $\epsilon_s{}^r$ and $e_s{}^r$, respectively, or

$$T^*{}_a{}^a = 3T^*, \qquad \epsilon_a{}^a = 3\epsilon, \qquad e_a{}^a = 3e. \tag{5}$$

As the Newtonian liquid possesses elasticity of volume, but not elasticity of form, T^* contains a part $(-p$, where p is called the hydrostatic pressure) which is a function of ϵ

$$p = F(\epsilon) \tag{6}$$

while $\epsilon'{}_s{}^r$ does not enter into the expression of $T^*{}_s{}^r$ in terms of strain. The hydrostatic pressure is not identical with the isotropic component of the stress tensor, but, as its name implies, only such part of it as remains for $e_s{}^r = 0$. (6) can also be written in the form

$$p = -3\kappa\epsilon \tag{6'}$$

where κ is the elastic bulk modulus and is itself a function of ϵ. We, therefore, write

$$T^*{}_s{}^r = (T - p)\delta_s{}^r + T'{}_s{}^r = (T + 3\kappa\epsilon)\delta_s{}^r + T'{}_s{}^r, \tag{7}$$

or

$$T^*{}_s{}^r - 3\kappa\epsilon\delta_s{}^r = T_s{}^r = T\delta_s{}^r + T'{}_s{}^r, \tag{8}$$

but we shall not deal in the present paper with (6). This is common to both solids and liquids and has received complete treatment by Murnaghan and Birch. Accordingly we write $T_s{}^r$ for $T^*{}_s{}^r - 3\kappa\epsilon\delta_s{}^r$ and consider in the following $T_s{}^r$ only.[3] The rheological equation (compare Hersey) of the Newtonian liquid in its most general form can now be written as follows:

$$T_s{}^r = R(\epsilon, e_s{}^r) \tag{9}$$

or, in view of (6),

$$T_s{}^r = f(p, e_s{}^r). \tag{10}$$

The elaboration of (10) can be carried out in two stages. In the first stage we write

$$T_s{}^r = f(e_s{}^r), \tag{11}$$

which means that the constants appearing in this equation will generally be functions of the hydrostatic pressure or of the density of the material. The dependence of the coefficient of viscosity upon the hydrostatic pressure is well known. However, just as in the classical hydrodynamics of viscous liquids the coefficient of viscosity is treated as a constant and its dependence upon p relegated to a secondary consideration, in the same way we shall in the present paper limit ourselves to (11). Our problem is to develop the function f in accordance with certain postulates to which we are led by physical observations.

3. The Murnaghan method. We first follow the way taken by Hamel and Murnaghan in accordance with which the rheological equation is derived from the two thermodynamical laws. For isothermal processes both are combined in the Gibbs-Helmholtz equation

$$\begin{aligned} \delta w &= \rho\delta\Phi + \rho\delta\psi \\ \delta\psi &\geq 0 \end{aligned} \tag{12}$$

where w is the strainwork per unit volume, Φ is the intrinsic[4] free energy-density and ψ is the bound energy-density. The second part of (12) expresses

[3] The tensor $T_s{}^r$ is the viscous resistance of the liquid in its most general sense.

[4] I. e., after deduction of the kinetic energy produced. This is effected in viscometry by means of the so-called kinetic energy correction.

the condition that while in the case of thermodynamic equilibrium ψ vanishes, in other cases it can only increase (compare Weissenberg). Generally, in a rheological process of every *real* material both functions Φ and ψ will be involved, but two *ideal* materials can be conceived. For the first ψ vanishes identically. This is the elastic solid, which has been dealt with by Murnaghan. For the second Φ vanishes in distortion or, more generally, in the absence of p. This is the viscous liquid with which we are dealing in the present paper. In the latter case we can introduce into (12) $\delta\Phi = 0$, when we get $\delta w = \rho\delta\psi$ and therefore

$$\delta w \geq 0. \tag{12'}$$

There is no complete analogy between Φ and ψ. While the former can be expressed as a function of mechanical variables with time not appearing explicitly, or $\partial\Phi/\partial t = 0$, this is generally not the case with regard to ψ. Nevertheless, as Rayleigh has shown, there exists a function $\chi = \chi(\dot{\psi})$ in the case of a simple Newtonian liquid, such that the hydrodynamic stress tensor can be expressed as a derivative of χ in analogy to the elastic stress tensor. In particular he found for such liquids

$$\chi = \rho\dot{\psi}/2 = \dot{w}/2. \tag{13}$$

He called the function χ the "dissipation function." When such a function exists, a theory of generalized Newtonian liquids in analogy to Murnaghan's theory of elastic solids is possible. *In the present section we shall postulate the existence of a dissipation function* and see where this leads us, while in the next section we shall drop this postulate and proceed by a more general method.

We therefore write in analogy to Murnaghan, but keeping in mind the simplification introduced by the fact that in our case the relative displacements are infinitesimal,

$$T_s^r = \partial\chi/\partial e_s^r. \tag{14}$$

The dissipation function is, in analogy to Φ, a function

$$\chi = \chi(I_1, I_2, I_3) \tag{15}$$

of the invariants of the tensor e_s^r, while the temperature does not appear explicitly because rheological processes are assumed to be isothermal.

From (14) we find in analogy to Murnaghan

$$T_s^r = \delta_s^r\partial\chi/\partial I_1 + (\delta_s^r I_1 - e_s^r)(\partial\chi/\partial I_2) + \zeta_s^r I_3\partial\chi/\partial I_3. \tag{16}$$

where the tensor ζ_s^r is defined by

$$\zeta_a^r e_s^a = \delta_s^r. \tag{17}$$

In order to apply (11) to the generalized Newtonian liquid, we calculate the strain-power per unit volume, which we denote by $\dot{w}$. This is equal to the trace of the inner product of the stress- and the velocity-strain tensor, or

$$\dot{w} = T_\beta{}^\alpha e_\alpha{}^\beta. \tag{18}$$

Making use of the relations

$$\begin{aligned} \delta_\beta{}^\alpha e_\alpha{}^\beta &= e_\alpha{}^\alpha = I_1 \\ e_\beta{}^\alpha e_\alpha{}^\beta &= I_1{}^2 - 2I_2 \\ \zeta_\beta{}^\alpha e_\alpha{}^\beta &= \delta_\alpha{}^\alpha = 3 \end{aligned} \tag{19}$$

we find from (18) and (16)

$$\dot{w} = I_1 \partial\chi/\partial I_1 + 2I_2 \partial\chi/\partial I_2 + 3I_3 \partial\chi/\partial I_3. \tag{20}$$

We now carry out the resolution of both tensors $T_s{}^r$ and $e_s{}^r$ in accordance with (1) and (3). This carries with it a resolution of the strain-power in accordance with the following equation,

$$T_\beta{}^\alpha e_\alpha{}^\beta = (T\delta_\beta{}^\alpha + T'_\beta{}^\alpha)(e\delta_\alpha{}^\beta + e'_\alpha{}^\beta) = 3Te + T'_\beta{}^\alpha e'_\alpha{}^\beta \tag{21}$$

into a volumetric component

$$\dot{w}_V = 3Te \tag{22}$$

and a deviatoric component

$$\dot{w}' = T'_\beta{}^\alpha e'_\alpha{}^\beta. \tag{23}$$

Considering the stress first, we find from (16) by means of

$$\delta_\alpha{}^\alpha = 3, \qquad e_\alpha{}^\alpha = I_1, \qquad \zeta_\alpha{}^\alpha I_3 = I_2, \tag{24}$$

$$T = T_\alpha{}^\alpha/3 = \partial\chi/\partial I_1 + \tfrac{2}{3} I_1 \partial\chi/\partial I_2 + I_2/3 \cdot (\partial\chi/\partial I_3) \tag{25}$$

and therefore

$$T'_s{}^r = \delta_s{}^r/3 \cdot (I_1 \partial\chi/\partial I_2 - I_2 \partial\chi/\partial I_3) - e_s{}^r \partial\chi/\partial I_2 + \zeta_s{}^r I_3 \partial\chi/\partial I_3. \tag{26}$$

Then we introduce the resolution of the strain. This affects the strain-invariants I_1, I_2 and I_3. They are replaced by

$$\begin{aligned} e_1 &= \tfrac{1}{3} I_1 \\ I'_2 &= I_2 - 3e^2 \\ I'_3 &= I_3 - I_2 e + 2e^3 \end{aligned} \tag{27}$$[5]

From the transformation equation

$$\partial\chi/\partial I_r = \partial\chi/\partial I'_\alpha \cdot \partial I'_\alpha/\partial I_r, \tag{28}$$

we get

[5] Note that I'_2 is not the same as Murnaghan's I'_2.

$$\begin{aligned} \partial\chi/\partial I_1 &= \tfrac{1}{3}\partial\chi/\partial e - 2e\partial\chi/\partial I'_2 - \tfrac{1}{3}(I'_2 - 3e^2)\partial\chi/\partial I'_3 \\ \partial\chi/\partial I_2 &= \partial\chi/\partial I'_2 - e\partial\chi/\partial I'_3 \\ \partial\chi/\partial I_3 &= \partial\chi/\partial I'_3 \end{aligned} \tag{29}$$

and our equations (25) and (26) become

$$T = \tfrac{1}{3}\partial\chi/\partial e \tag{30}$$

$$\begin{aligned} T'_s{}^r = &- \delta_s{}^r[(I'_2 + 3e^2)/3 \cdot \partial\chi/\partial I'_3] - e'_s{}^r[\partial\chi/\partial I'_2 - e\partial\chi/\partial I'_3] \\ &+ \zeta_s{}^r[(I'_3 + I'_2 e + e^3)\partial\chi/\partial I'_3], \end{aligned} \tag{31}$$

while $\dot{w}_V$ and $\dot{w}'$ are from (22) and (23)

$$\dot{w}_V = e\partial\chi/\partial e \tag{32}$$

$$\dot{w}' = 2I'_2\partial\chi/\partial I'_2 + 3I'_3\partial\chi/\partial I'_3. \tag{33}$$

Murnaghan's next step is to consider ϕ as a function $\phi(\epsilon_s{}^r)$ of the strain components through the invariants and to expand it into a power series, neglecting terms of orders of magnitude greater than an agreed upon order. Elastic strains of even very deformable materials are small in comparison with the velocity-strains of viscous liquids which find their limit in turbulence only. A power-series development will therefore not carry us far. We should, however, naturally expect the lowest approximation of the generalized Newtonian liquid to yield the expressions for the simple Newtonian liquid. If we put

$$\chi = ae + be^2 + cI'_2 + \cdots \tag{34}$$

we find

$$\begin{aligned} T = a/3 + 2be/3 + \cdots; \qquad T'_s{}^r = &- ce'_s{}^r + \cdots; \\ \dot{w} = ae + 2be^2 + 2cI'_2 + \cdots. \end{aligned} \tag{35}$$

For the case of rest this would give $T = a/3$, i. e., a hydrostatic stress, the consideration of which we have excluded. We therefore put $a = 0$ which gives $\chi = \dot{w}/2$ in accordance with Rayleigh's expression for the dissipation function (compare (13)). This, evidently, has a wider range than the classical viscous liquid, because for the latter, in order to get Stokes' expression for the stress, we have, in addition, to put $b = 0$ which gives $T = 0$ and $T_s{}^r = 2\eta(e_s{}^r - e\delta_s{}^r)$ with η, the coefficient of viscosity, $= -c/2$.

4. Application of a tensor analysis method.[6] The results of the previous chapter are valid only in the restricted case when there exists a dissipation function. In the present chapter we drop this restriction and proceed by pure

[6] This treatment was suggested to me by Prof. Giulio Racah of the Hebrew University, Jerusalem.

tensor analysis of (11). Starting from this equation we note that on the left side there stands a mixed tensor of rank two. Then what we may call with Eddington the *principle of generalized dimensions* requires that all terms on the right side shall also be mixed tensors of rank two. The right side can therefore consist only of sums of mixed tensors of rank two multiplied by scalars and of inner products of such tensors which again are reduced to tensors of rank two. The general term of a development of the function f will therefore be of the form $e_\alpha{}^r e_\beta{}^\alpha e_\gamma{}^\beta \cdots e_s{}^\lambda \cdot f(I)$, where $f(I)$ is a function of the three invariants; and we can therefore write

$$(36) \quad T_s{}^r = f_0(I)\delta_s{}^r + f_1(I)e_s{}^r + f_2(I)e_\alpha{}^r e_s{}^\alpha + f_3(I)e_\alpha{}^r e_\beta{}^\alpha e_s{}^\beta + \cdots.$$

This would mean an infinite number of such terms. However, in view of the Cayley-Hamilton equation of matrix theory, the following relation holds good,

$$(37) \quad e_\alpha{}^r e_\beta{}^\alpha e_s{}^\beta = \delta_s{}^r I_3 - e_s{}^r I_2 + e_\alpha{}^r e_s{}^\alpha I_1$$

and therefore

$$(38) \quad \begin{aligned} e_\alpha{}^r e_\beta{}^\alpha e_\gamma{}^\beta e_s{}^\gamma &= \delta_\alpha{}^r e_s{}^\alpha I_3 - e_\alpha{}^r e_s{}^\alpha I_2 + e_\alpha{}^r e_\beta{}^\alpha e_s{}^\beta I_1 \\ &= \delta_s{}^r I_1 I_3 + e_s{}^r(I_3 - I_1 I_2) + e_\alpha{}^r e_s{}^\alpha(I_1{}^2 - I_2) \end{aligned}$$

and similarly in respect of higher terms. This enables us to write

$$(39) \quad T_s{}^r = F^*{}_0(I)\delta_s{}^r + F_1(I)e_s{}^r + F_2(I)e_\alpha{}^r e_s{}^\alpha$$

and also

$$(40) \quad T_s{}^r = F^*{}_0(I')\delta_s{}^r + F_1(I')e'_s{}^r + F_2(I')e'_\alpha{}^r e'_s{}^\alpha$$

where $F^*{}_0$, F_1 and F_2 are functions different from those of (39).

Resolving the stress, we get

$$(41) \quad \begin{aligned} T &= F^*{}_0(I') - 2I'_2/3 \cdot F_2(I') = F_0(I') \\ T'_s{}^r &= F_1(I')e'_s{}^r + F_2(I')(e'_\alpha{}^r e'_s{}^\alpha + 2I'_2/3 \cdot \delta_s{}^r) \end{aligned}$$

where we have made use of the relation

$$(42) \quad e'_\beta{}^\alpha e'_\alpha{}^\beta = -2I'_2.$$

In accordance with (22) and (23) we find

$$(43) \quad \dot{w}_V = 3eF_0(I'),$$

$$(44) \quad \dot{w}' = -2I'_2 F_1(I') + 3I'_3 F_2(I'),$$

where we have made use of the relation

$$(45) \quad e'_\beta{}^\alpha e'_\gamma{}^\beta e'_\alpha{}^\gamma = 3I'_3.$$

If we compare (41) with (30) and (31), making use of the relation

$$e_a{}^r e_s{}^a = \zeta_s{}^r I_3 - \delta_s{}^r I_2 + e_s{}^r I_1 \tag{46}$$

which is analogous to (37), we see that a dissipation function exists if F_0, F_1 and F_2 can be expressed as the derivatives of a single function as follows:

$$\begin{aligned} F_0(I') &= \partial\chi/\partial I'_1 \\ F_1(I') &= -\partial\chi/\partial I'_2 \\ F_2(I') &= \partial\chi/\partial I'_3. \end{aligned} \tag{47}$$

Elimination of χ from (47) gives

$$\begin{aligned} \partial F_0/\partial I'_2 + \partial F_1/\partial I'_1 &= 0 \\ \partial F_1/\partial I'_3 + \partial F_2/\partial I'_2 &= 0 \\ \partial F_2/\partial I'_1 - \partial F_0/\partial I'_3 &= 0. \end{aligned} \tag{48}$$

5. Expression of strain through stress. It is obvious that we can proceed quite analogously by starting from the reverse of (11), expressing $e_s{}^r$ as a function of $T_s{}^r$. We get instead of (41), (43) and (44) the following:

$$\begin{aligned} \epsilon &= \mathfrak{F}_0(T') \\ \epsilon'_s{}^r &= \mathfrak{F}_1(T')T'_s{}^r + \mathfrak{F}_2(T')(T'_a{}^r T'_s{}^a + 2T'_2/3 \cdot \delta_s{}^r), \end{aligned} \tag{49}$$

$$\begin{aligned} \dot{w}_V &= 3T\mathfrak{F}_0(T') \\ \dot{w}' &= -2T'_2\mathfrak{F}_1(T') + 3T'_3\mathfrak{F}_2(T') \end{aligned} \tag{50}$$

where the meaning of T', T'_2 and T'_3 is obvious. If we introduce $\epsilon'_s{}^r$ from the second of (49) into the expression for $T'_s{}^r$ (second of (41)), taking into account the stress analogies to (37) and (38), we find

$$\begin{aligned} T'_s{}^r = {} & [2I'_2/3 \cdot F_2 + 2T'_2/3 \cdot \mathfrak{F}_2F_1 + (2T'_2/3)^2\mathfrak{F}_2{}^2F_2 \\ & + 2\mathfrak{F}_1\mathfrak{F}_2F_2(T_3 - TT_2 + 2T^3)]\delta_s{}^r \\ & + [\mathfrak{F}_1F_1 + 4T'_2/3 \cdot \mathfrak{F}_1\mathfrak{F}_2F_2 + 2\mathfrak{F}_1\mathfrak{F}_2F_2(3T^2 - T_2) \\ & + \mathfrak{F}_2{}^2F_2(T_3 - TT_2 + 2T^3)]T'_s{}^r + [\mathfrak{F}_2F_1 + \mathfrak{F}_1{}^2F_2 \\ & + 4T'_2/3 \cdot \mathfrak{F}_2{}^2F_2 + \mathfrak{F}_2{}^2F_2(3T^2 - T_2)]T'_a{}^r T'_s{}^a. \end{aligned} \tag{51}$$

Considering that the $\delta_s{}^r$, $T'_s{}^r$, $T'_a{}^r T'_s{}^a$ stand for the zero, first and second powers in the stress components, this requires that the first and third expressions within brackets [ ] must vanish and the second must be equal to unity. Introducing furthermore

$$\begin{aligned} T_2 &= T'_2 + 3T^2 \\ T_3 &= T'_3 + TT'_2 + T^3 \end{aligned} \tag{52}$$

we find the three equations

$$
\begin{aligned}
&T'_2\mathfrak{J}_2F_1 + [2T'^2_2/3\cdot\mathfrak{J}_2 + 3\mathfrak{J}_1T'_3]\mathfrak{J}_2F_2 + I'_2F_2 = 0\\
&\mathfrak{J}_1F_1 + [\mathfrak{J}_2T'_3 - 2T'_2/3\cdot\mathfrak{J}_1]\mathfrak{J}_2F_2 = 1\\
&\mathfrak{J}_2F_1 + [\mathfrak{J}_1^2 + T'_2/3\cdot\mathfrak{J}_2^2]F_2 = 0
\end{aligned}
\tag{53}
$$

from which the F_1, F_2 and I'_2F_2 can be expressed through the stress invariants. There are obviously analogous equations by which the $\mathfrak{J}_1$, $\mathfrak{J}_2$ and $T'_2\mathfrak{J}_2$ can be expressed through the strain invariants.

6. Physical interpretation. The physical interpretation of our theory will consist in assigning physical meanings to the three functions F_0, F_1, F_2 and $\mathfrak{J}_0$, $\mathfrak{J}_1$, $\mathfrak{J}_2$. For this purpose we consider certain special cases of stress and strain which constitute simple viscometric tests.

(i) The fundamental viscometric test, i e., the one which from Newton onwards served to define the coefficient of viscosity, is based upon a laminar plane deformation of the type

$$
e_s{}^r = e'_s{}^r = \begin{Vmatrix} 0 & e_{xy} & 0 \\ e_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{54}
$$

for which

$$
e = 0; \qquad I_2 = I'_2 = -e_{xy}{}^2; \qquad I_3 = I'_3 = 0 \tag{55}
$$

and

$$
e_a{}^r e_s{}^a = e'_a{}^r e'_s{}^a = e^2{}_{xy} \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}. \tag{56}
$$

This gives in accordance with (41)

$$
T = F_0; \quad T'_s{}^r = F_1 e_{xy} \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} + F_2 e^2{}_{xy}/3 \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{Vmatrix}, \tag{57}
$$

while (43) and (44) give

$$
\dot{w} = \dot{w}' = 2e^2{}_{xy}F_1. \tag{58}
$$

Dissipation of energy, therefore, comes in this case from F_1 only. In the case of the simple Newtonian liquid the dissipated power in simple shear is $= 4\eta e^2{}_{xy}$.[7] We follow Einstein in defining the coefficient of viscosity by equating the strain-power in our case with the strain-power of the similar flow of a simple Newtonian liquid. We accordingly get

[7] One should note that our e_{xy} is one half of what is usually denoted by e_{xy}.

$$F_1(0, I'_2, 0) = 2\eta. \tag{59}$$

(ii) The simple cubical dilatation

$$e_s{}^r = e\delta_s{}^r; \qquad e'_s{}^r = 0; \qquad I'_2 = I'_3 = 0 \tag{60}$$

gives

$$T_s{}^r = F_0\delta_s{}^r; \qquad \dot{w} = \dot{w}_v = 3eF_0. \tag{61}$$

In primitive cases rheological coefficients are defined from stress-strain relations. This method can be applied if the ratio of stress to strain is a scalar and could therefore not be applied in case (i). However, in our present case we get

$$T_s{}^r/e_s{}^r = F_0(e, 0, 0)/e = \eta_v, \tag{62}$$

where η_v may be called the *coefficient of volume-viscosity*. The theoretical existence of such a second coefficient of viscosity was pointed out by Reiner. Recently Tisza has confirmed its actual existence from experiment.

Analogously, if we start from the simple isotropic stress $T^*{}_s{}^r = T^*\delta_s{}^r$, we arrive at a *coefficient of volume-fluidity* defined by

$$\phi_v = \mathfrak{F}_0(T, 0, 0)/T. \tag{63}$$

(iii) The stress-analogy to (i) is the simple shearing stress in vacuo

$$T^*{}_s{}^r = T^{*\prime}{}_s{}^r = \begin{Vmatrix} 0 & T^*_{xy} & 0 \\ T^*_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{64}$$

so that in accordance with (8)

$$\begin{aligned} &T_s{}^r = T^*{}_s{}^r - 3\kappa\epsilon\delta_s{}^r \\ &T = -3\kappa\epsilon; \quad T'_s{}^r = \end{aligned} \begin{Vmatrix} 0 & T^*_{xy} & 0 \\ T^*_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{65}$$

and

$$T'_2 = -T^{*2}_{xy}; \qquad T'_3 = 0 \tag{66}$$

Expressing the strain through the stress we find from (49)

$$\begin{aligned} &e = \mathfrak{F}_0 \\ &e'_s{}^r = \mathfrak{F}_1 T^*_{xy} \end{aligned} \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} + \mathfrak{F}_2 T^{*2}_{xy}/3 \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{Vmatrix}. \tag{67}$$

We now arrange our experiment so that we keep T^*_{xy} constant and start our observations *after a steady state has been reached.* We should then, in accordance with (67), observe, besides a continuous shearing of the liquid the rate of which is $\mathfrak{F}_1 T^*_{xy}$, also a continuous expansion (or contraction) the rate

of which is $\mathfrak{F}_0$ and a continuous lengthening (or shortening) in the direction normal to the plane of shear (in our case the z-direction) the rate of which is $\mathfrak{F}_2 T^{z*}_{xy}$. The second and the third phenomenon are against experience. There may be a cubical dilatation when T^*_{xy} is applied—but this raises an elastic resistance which ultimately stops it and when the steady state is reached the rate of dilatation will have become zero. And there cannot be any extension at all in the z-direction, because, the liquid being devoid of rigidity, there would be nothing to stop it and it would go on indefinitely—which has never been observed. Therefore we must have

$$\begin{aligned} \mathfrak{F}_0(0, T'_2, 0) &= 0 \\ \mathfrak{F}_2(0, T'_2, 0) &= 0 \end{aligned} \tag{68}$$

which reduces (67) to

$$e_s{}^r = \mathfrak{F}_1 T^*_{xy} \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}. \tag{69}$$

From (69) we arrive at the coefficient of fluidity ϕ

$$\phi = 2e_{xy}/T^*_{xy} = \mathfrak{F}_1(0, T'_2, 0). \tag{70}$$

To express the stress through the strain we use (53) from which we find

$$F_2(0, I'_2, 0) = 0; \qquad F_1(0, I'_2, 0) = 1/\mathfrak{F}_1(0, T'_2, 0) \tag{71}$$

so that

$$T^*_s{}^r = (3\kappa\epsilon + F_0)\delta_s{}^r + T^*_{xy} \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}. \tag{72}$$

This brings us back to our starting point (64) and therefore

$$3\kappa\epsilon + F_0 = 0 \tag{73}$$

or

$$\epsilon_a{}^a = 3\epsilon = -F_0(0, I'_2, 0)/\kappa. \tag{74}$$

While, therefore, in the simple Newtonian liquid a simple shearing stress produces a simple shearing strain continuously increasing at a constant rate—and nothing else; in the generalized Newtonian liquid *it will*, in addition, *produce a cubical dilatation* the magnitude of which depends upon a rheological coefficient

$$\delta = F_0(0, I'_2, 0)/I'_2 \tag{75}$$

which we may call the "coefficient of dilatancy" δ (not to be confused with Kronecker's $\delta_s{}^r$), so that

(76) $$\epsilon_G{}^a = -(\delta/\kappa)\cdot I'_2.$$

Conversely, *if the liquid is contained between rigid walls* as in most viscometers, so that the dilatation is prevented, *a pressure is exerted upon the walls* which is, in accordance with (57),

(77) $$T = F_0 = \delta I'_2.$$

7. Conclusions. For the purpose of our present investigations we may assume that F_2 vanishes identically and that F_0 and F_1 are independent of I'_3.[8] This results in a special case of the generalized Newtonian liquid which we may call the *Reynolds Liquid.* The rheological equation of the Reynolds liquid is accordingly

(i) $$T^*{}_s{}^r = (3\kappa\epsilon + \eta_\nu e + \delta I'_2)\delta_s{}^r + 2\eta e'_s{}^r$$

where $T^*{}_s{}^r$ is the total stress—elastic and viscous—3ϵ the cubical dilatation, e the isotropic component and $e'_s{}^r$ the deviator of the velocity-strain and κ, η, η_ν and δ are four rheological coefficients, viz. the bulk modulus, the ordinary (Newtonian) coefficient of viscosity, the coefficient of volume viscosity and the coefficient of dilatancy, respectively. These are, to a first approximation, rheological constants with temperature and density as parameters. The total strain-power is

(ii) $$\dot{w} = (3\kappa\epsilon + \eta_\nu e + \delta I'_2)3e - 4\eta I'_2.$$

The contributions from free and bound energy can be separated by means of the condition that for the latter $\dot{w} \geq 0$. We accordingly get

(iii)[9] $$\rho\dot{\phi} = 9\kappa\epsilon e + 3\delta I'_2 e,$$
$$\rho\dot{\psi} = 3\eta_\nu e^2 - 4\eta I'_2.$$

If both coefficients of viscosity are constants, or if the coefficient of volume viscosity is a function of e only and the Newtonian coefficient of viscosity a function of I'_2 only, there exists a dissipation function

(iv) $$\chi = (3/2)\cdot e^2\eta_\nu - 2\eta I'_2 = \rho\dot{\psi}/2.$$

Dilatancy as defined here does not contribute to the dissipation of energy. It emerges from (i) in two forms, viz.:

(i) A deviator of velocity strain for which ϵ and e vanish produces an isotropic pressure $= \delta I'_2$.

[8] I'_3 and F_2 appear in what Trouton called "viscous traction." This will be dealt with in a separate communication.

[9] I'_2 is always negative.

(ii) A deviator of stress for which T^* vanishes produces a cubical dilatation which ultimately (i. e., when $e = 0$) $= -(\delta/\kappa) \cdot I'_2$.

It is the latter phenomenon which Reynolds described as "dilatancy."

I take occasion to thank Professor Racah for the interest which he took in the preparation of the paper.

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ALGEBRAISATION OF PLANE ABSOLUTE GEOMETRY.*

By Y. Why Tschen.

Introduction. We give in this paper a simple method, based on the axioms given by K. Reidemeister and F. Bachmann,[1] of deriving the algebraisation of plane absolute geometry up to the existence of a quadratic form. On studying the paper AG just mentioned and the well-known paper of J. Hjelmslev, "*Neue Begründung der ebenen Geometrie*," [2] we thought it might be possible on the one hand to generalize Hjelmslev's halfrotations to those with an improper point as center and on the other hand to find out some metric properties of the generalized halfrotations. Indeed we have only to formulate Hjelmslev's halfrotations in a slightly different manner to obtain the generalization. A very useful lemma for us concerning the metric properties of the halfrotations may be stated as follows: "*If a halfrotation* $H_{\mathfrak{d}}$ *which belongs to the rotation* $\mathfrak{d}$ [3] *transforms a line* l *into a line* l', *then the halfrotation* $H_{\mathfrak{d}^{-1}}$ *which belongs to the inverse-rotation* $\mathfrak{d}^{-1}$ *will transform any line orthogonal to* l' *into a line orthogonal to* l." This lemma can easily be proved.

To prove Desargues' and Pascal's theorems we shall not use Hessenberg's method of "involutions of equal angles" [4] by which Hjelmslev proved Pascal's theorem containing only proper elements. Nor do we need the artificial concept of "cinematic space" (AG § 4) of Reidemeister and Bachmann. We shall first prove a special case of Desargues' theorem: the Desargues' configuration with any proper point P as perspective center and its polar p as perspective axis. The same theorem about the configurations with any proper point $Q \neq O$ as center but always with the polar o of the same proper point O as axis will be reduced to the special case just mentioned by using our lemma. Finally the most general Desargues' configuration which contains the line o and eventually improper points and lines will be transformed by half-

* Received November 22, 1944.

[1] F. Bachmann, "Eine Begründung der absoluten Geometrie in der Ebene," *Mathematische Annalen*, vol. 113, pp. 424-451. This paper will be referred to as AG.

F. Bachmann and K. Reidemeister, "Die metrische Form in der absoluten und der elliptischen Geometrie," *Mathematische Annalen*, vol. 113, pp. 748-765.

[2] *Mathematische Annalen*, vol. 64, pp. 449-474. We shall refer to this paper as NG.

[3] $\mathfrak{d}$ must be non-involutoric. The center of $\mathfrak{d}$ may be proper or improper.

[4] G. Hessenberg, *Grundlagen der Geometrie* (1930), p. 79.

rotations with the center O[5] into those configurations which, save for o, contain only proper elements. Thus all Desargues' configurations containing the fixed line o exist. Hence, according to a well-known theorem, Desargues' theorem is generally true in the projective plane.

We prove the special case of Desargues' theorem mentioned above by showing the existence of perspective collineations[6] with P as center and p as axis. We obtain these collineations as the product transformations of halfrotations and their inverses with the center P.[7] From the fact that halfrotations with the same proper point as center are commutative[8] we incidentally obtain Pascal's theorem (which is equivalent to the commutativity of perspective collineations with the same point as center).

We prove the existence of a polarity in our geometry with the help of our lemma. The involution of the right angles at a proper point can be induced by the product transformation of a halfrotation and an inverse halfrotation. As the latter are linear transformations the involution is projective and, consequently, the polarity is a linear correlation. Thus we obtain a quadratic form which is left invariant by the motions in the plane.

Our whole development is based on the axioms given in AG § 1 and their immediate consequences in AG § 2. We shall also use the results in AG, § 3 concerning improper points. We consider, before we start, that the improper points have been introduced into the plane. So if we say "points" we mean proper or improper points. But we shall have nothing to do with the improper lines, before we have introduced them. Although we follow the same principle as in AG and NG to introduce the improper lines, our method of representing the totality of points on an improper line is quite different; and our proof of the existence of a polarity depends partially upon the possibility of such representations.

We sketch in the last section a foundation of hyperbolic geometry by adding to the axioms of plane absolute geometry a new axiom, which guarantees the existence of boundary points. This new axiom does not involve any notions not previously used. Properties of boundary points are studied and improper lines introduced in a more explicit way.

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[5] NG, p. 471.

[6] See for reference Veblen and Young, *Projective Geometry* I (1916), p. 72. The existence of such collineations with two homologous arbitrarily given points is equivalent to the existence of all Desargues' configurations with center P and axis p.

[7] We come to this product transformation by considering the Cayley formula $S = (1 + R)/(1 - R)$ which connects an orthogonal matrix R with a skew matrix S. A half-rotation can be represented as $\frac{1}{2}(1 + R)$ and the product of two skew matrices is a diagonal matrix, which represents a perspective collineation.

[8] AG, Theorem 63; NG, p. 466.

The question as to what extent and under what conditions the semi-euclidean and semi-noneuclidean geometries exist is not studied in this paper.

Notations. Capital Latin letters denote points, small Latin and Greek letters denote lines as well as reflexions. A point, which is common to two different lines a and b will also be denoted by (a, b) and a line, which is determined by two different points A, B of which at least one is proper will be denoted by $< A, B >$. If $a \perp b$, then (a, b) is a proper point.

I. Halfrotations.

Definition 1. A motion in the plane, which can be represented by $\alpha \cdot a \cdot \alpha'$ with $\alpha \perp a$, $\alpha' \perp a$ is called a sliding-reflexion. A sliding-reflexion with $\alpha = \alpha'$ is simply a reflexion.

THEOREM 1. *Let a motion $\mathfrak{M}$ be a sliding-reflexion represented by $\alpha \cdot a \cdot \alpha'$. Then the mid-point of a proper point P and its image $\mathfrak{M}(P)$ always lies on a.*

Proof. If P is on a, the theorem is trivial. Let P be not on a, and let $\bar{a}$ be the line passing through P and orthogonal to a. Then we have $\alpha \cdot \alpha' = \bar{\alpha} \cdot \bar{\alpha}'$ with $\bar{\alpha} \perp a$, $\bar{\alpha}' \perp a$. By the motion $\mathfrak{M}$ the point P will be transformed into its image $\mathfrak{M}(P) = a \cdot \bar{a} \cdot \bar{\alpha}'(P) = \bar{a} \cdot a \cdot \bar{\alpha}'(P) = a \cdot \bar{\alpha}'(P)$ for P lies on $\bar{a}$ and $\bar{a}(P) = P$. We see immediately that $(a, \bar{\alpha}')$ is the mid-point of P and $\mathfrak{M}(P)$.

THEOREM 2. *If a sliding-reflexion can be represented both by $\alpha \cdot a \cdot \alpha'$ and by $b \cdot \beta \cdot \beta'$ with $\alpha \perp a$, $\alpha' \perp a$; $\beta \perp b$, $\beta' \perp b$ then the two lines a and b are not distinct and $\alpha \cdot \alpha' = \beta \cdot \beta'$.*

Proof. We have, in the previous theorem, given the line a a geometrical meaning as the locus of midpoints. Hence $a = b$ and $\alpha \cdot \alpha' = \beta \cdot \beta'$.

Definition 2. The line a is called the axis of the sliding-reflexion $\alpha \cdot a \cdot \alpha'$.

THEOREM 3. *The product of any three reflexions is a sliding-reflexion.*

Proof. If the three lines pass through a point then their product is a reflexion. Otherwise we may choose three lines a', b' and c' so that $a' \cdot b' \cdot c' = a \cdot b \cdot c$ and (a', b') is a proper point. Let $\bar{b}$ be the perpendicular to c' which passes through (a', b') and $\bar{a} \cdot \bar{b} = a' \cdot b'$. Then we have $\bar{a} \cdot \bar{b} \cdot c' = \bar{a} \cdot (\bar{b} \cdot c') = \rho \cdot (d \cdot \bar{\rho})$ where we denote by d the line which passes through $(\bar{b}, c')$ and is orthogonal to $\bar{a}$; $\rho \equiv \bar{a}$, and ρ' is the line orthogonal to d at $(\bar{b}, c')$. Hence $a \cdot b \cdot c$ is the sliding-reflexion $d \cdot \rho \cdot \rho'$.

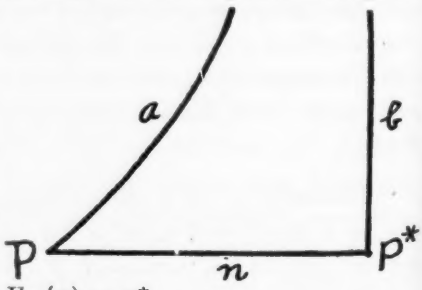
Definition 3. Halfrotations. Let $\mathfrak{d}$ be a rotation in the plane with $\mathfrak{d}^2 \neq 1$. The center of $\mathfrak{d}$ may be proper or improper. For each line x in the plane we form the product $x \cdot \mathfrak{d}$ which can be represented as a sliding-reflexion $y \cdot \eta \cdot \eta'$ with $\eta \perp y$, $\eta' \perp y$. The axis y is uniquely determined by x and $\mathfrak{d}$. The line transformation $x \to y$ is called the halfrotation $H_{\mathfrak{d}}$. In case $\mathfrak{d} = a \cdot b$ we use also the symbol $H_{a,b}$ instead of $H_{\mathfrak{d}}$.

THEOREM 4. *If $x \neq x'$, then $H_{\mathfrak{d}}(x) \neq H_{\mathfrak{d}}(x')$.*

Proof. In case $x \cdot \mathfrak{d} = y \cdot \eta \cdot \eta'$ and $x' \cdot \mathfrak{d} = y \cdot \bar{\eta} \cdot \bar{\eta}'$ with $x \neq x'$ we would have $x \cdot x' = (x \cdot \mathfrak{d})(\mathfrak{d}^{-1} \cdot x') = (y \cdot \eta \cdot \eta')(\bar{\eta}' \cdot \bar{\eta} \cdot y) = \eta \cdot \eta' \cdot \bar{\eta}' \cdot \bar{\eta}$ with the η's all orthogonal to y. Hence $x \perp y$, $x' \perp y$ and $x \cdot \mathfrak{d} = y \cdot \eta \cdot \eta'$ would imply $\mathfrak{d} = x \cdot y \cdot \eta \cdot \eta' = y \cdot \eta''$ with $\eta'' \perp y$, which means $\mathfrak{d}^2 = 1$. As we have always $\mathfrak{d}^2 \neq 1$ in $H_{\mathfrak{d}}$, so $H_{\mathfrak{d}}(x) \neq H_{\mathfrak{d}}(x')$.

THEOREM 5. *Let P be any proper point and let P^* be the mid-point of P and $\mathfrak{d}(P)$, $\mathfrak{d}^2 \neq 1$. Then the lines passing through P will be transformed $(1,1)$ to the lines through P^* by the halfrotation $H_{\mathfrak{d}}$.*

Proof. Let $\mathfrak{d} = a \cdot b$. We can assume that the line a passes through P. We denote by n the line that passes through P and is perpendicular to b, $n \neq a$. Then $P^* = (n, b)$ and P^* is a proper point.



Now let p be any line on P. We form $p \cdot \mathfrak{d}$ by putting $p \cdot a = \pi \cdot n$ and we have $p \cdot \mathfrak{d} = (\pi \cdot n) \cdot b = \pi \cdot (n \cdot b) = \pi \cdot p^* \cdot \pi'$ with $p^* \perp \pi$ and p^* passing through P^* and $p^* \perp \pi'$ at P^*. Hence $H_{\mathfrak{d}}(p) = p^*$.

Conversely let p^* be any line on P^*. π' denotes the line $\perp$ to p^* at P^* and π denotes the line which passes through P and is orthogonal to p^*. Then we get $p^* \cdot \pi \cdot \pi' = \pi \cdot n \cdot b$. Let $p = \pi \cdot n \cdot a$; then p is the line which is transformed by $H_{\mathfrak{d}}$ to p^*.

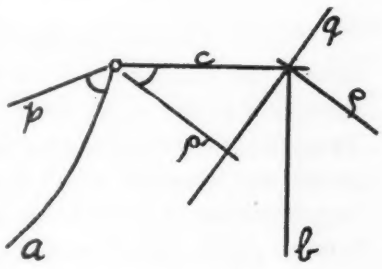
COROLLARY. *$H_{\mathfrak{d}}$ induces a point transformation among the proper points. If the pencil of lines at the proper point P is transformed into the pencil of lines at the proper point P^*, then we define $H_{\mathfrak{d}}(P) = P^*$.*

THEOREM 6. *$H_{\mathfrak{d}}$ transforms every proper point into a proper point and every line into a line. If $X \neq Y$, then $H_{\mathfrak{d}}(X) \neq H_{\mathfrak{d}}(Y)$, and if $x \neq y$ then $H_{\mathfrak{d}}(x) \neq H_{\mathfrak{d}}(y)$. The fact that the proper point X lies on the line x implies that $H_{\mathfrak{d}}(X)$ lies on $H_{\mathfrak{d}}(x)$.*

THEOREM 7. (Lemma on the metric properties of halfrotations.) *Let* $H_{\mathfrak{d}}(p) = q$. *Then the halfrotation* $H_{\mathfrak{d}^{-1}}$ *transforms every line orthogonal to* q *into a line orthogonal to* p *and if a perpendicular to* p *is the image of a line in the halfrotation* $H_{\mathfrak{d}^{-1}}$, *then the latter is a perpendicular to* q.

Proof. Suppose that p and q both pass through the center of $\mathfrak{d}$; then $\mathfrak{d} = p \cdot q$, $\mathfrak{d}^{-1} = q \cdot p$. According to the definition of $H_{\mathfrak{d}^{-1}}$ any line $\rho \perp q$ will be transformed into the line which passes through (q, ρ) and is orthogonal to p.

We now consider the case where p and q do not pass through the center of $\mathfrak{d}$. Let ρ be any perpendicular to q and let $(q, \rho) = Q$. We can assume that $\mathfrak{d} = a \cdot b$ with b on Q, and $b \neq q$. Let c be the line $\perp b$ at Q; then $c \neq a$, as $\mathfrak{d}^2 \neq 1$. The assumption $H_{\mathfrak{d}}(p) = q$ implies the existence of a perpendicular ρ' to q so that $p \cdot \mathfrak{d} = p \cdot a \cdot b = q \cdot \rho' \cdot \rho$. Since $q \cdot \rho = b \cdot c$, we have $p \cdot a \cdot b = q \cdot \rho' \cdot \rho = \rho' \cdot c \cdot b$. Hence $p \cdot a = \rho' \cdot c$ or $p \cdot \rho' = a \cdot c$. Therefore the lines p and ρ' both pass through the proper or improper point (a, c). $\rho \cdot \mathfrak{d}^{-1} = \rho \cdot b \cdot a = q \cdot c \cdot a = q \cdot \rho' \cdot p$. We denote (q, ρ') by Q' and the line, which passes through Q' and is orthogonal to p, by π; then we have $q \cdot \rho' \cdot p = \pi \cdot p' \cdot p$ with $p' \perp \pi$ at Q'. Hence $\rho \cdot \mathfrak{d}^{-1} = \pi \cdot p' \cdot p$ with $p \perp \pi$, $H_{\mathfrak{d}^{-1}}(\rho) = \pi$.



Finally let $\pi \perp p$ and suppose that there exists a line x with $H_{\mathfrak{d}^{-1}}(x) = \pi$. We take a proper point X on x and let $H_{\mathfrak{d}^{-1}}(X) = Y$. Then Y lies on π. The perpendicular to q which passes through X must be transformed by $H_{\mathfrak{d}^{-1}}$ into the perpendicular to p, which passes through Y. The latter is π. Hence $x \perp q$.

Definition 4. A halfrotation $H_{\mathfrak{d}}$ is called ordinary, if the center of $\mathfrak{d}$ is a proper point. This point will be called the center of $H_{\mathfrak{d}}$.

THEOREM 8. *The line transformation in the plane, defined by an ordinary halfrotation with the center* O, *can be stated as follows:*[9]

1° *If* g *passes through* O, $g \cdot g_1 = \mathfrak{d}$, *then* $H_{\mathfrak{d}}(g) = g_1$.

2° *If* g *does not pass through* O, *then let* $q \perp g$ *where* q *passes through* O, $q \cdot q_1 = \mathfrak{d}$. *The line* g_1 *which passes through* (q, g) *and is orthogonal to* q_1 *is the image of* q *in* $H_{\mathfrak{d}}$.

[9] This was used in AG as the definition of halfrotations. The proof of this is immediate.

From this it follows immediately, that any line orthogonal to a line a on O will be transformed to a line $\perp b$, if $H_\mathfrak{H}(a) = b$.

THEOREM 9. *An ordinary halfrotation $H_\mathfrak{H}$ transforms a proper point into a proper point, but the image of an improper point in $H_\mathfrak{H}$ may be improper or proper. The point transformation $H_\mathfrak{H}$ of all the points in the plane is a $(1, 1)$ transformation. Hence an ordinary $H_\mathfrak{H}$ always possesses an inverse point transformation $H_\mathfrak{H}^{-1}$. If $H_\mathfrak{H}(x) = x'$ then the totality of points on x will be transformed into the totality of points on x'. In case there exists for a given line y a line y', which has the line y as its image in $H_\mathfrak{H}$, we shall write $H_\mathfrak{H}^{-1}(y) = y'$. Then the totality of points on y will be transformed by $H_\mathfrak{H}^{-1}$ into the totality of points on y'.*

Proof. The fact that all the lines which pass through the same improper point X are transformed by an ordinary $H_\mathfrak{H}$ into the lines all of which pass through a proper or an improper point Y is proved in AG, Theorem 58. From this it follows that an ordinary halfrotation defines a transformation of proper and improper points $H_\mathfrak{H}(X) = Y$.[10] We shall prove that this point transformation is $(1, 1)$ in the plane. With regard to the incidence relations between points (proper and improper) and lines an ordinary halfrotation is a collineation. Two distinct points P and Q must have distinct images. For proof we take a proper point $R \neq P$, $R \neq Q$, not on $< P, Q >$, and the lines $< R, P > = p$, $< R, Q > = q$. Let $P = (p, p')$, $Q = (q, q')$. Then the lines p' and q' do not pass through R. $H_\mathfrak{H}(p) \neq H_\mathfrak{H}(q)$. $H_\mathfrak{H}(p')$ does not pass through $H_\mathfrak{H}(R)$. In case $H_\mathfrak{H}(P) = H_\mathfrak{H}(Q)$, all the images $H_\mathfrak{H}(p)$, $H_\mathfrak{H}(q)$, $H_\mathfrak{H}(q')$ and $H_\mathfrak{H}(p')$ must pass through the same point $H_\mathfrak{H}(R)$, which is impossible.

Now let P be any point of the plane. As $H_\mathfrak{H}$ always transforms proper points into proper points, so we can arbitrarily choose two proper points P_1 and P_2 in the plane which are images of proper points P^*_1 and P^*_2 respectively, and which do not lie on the same line passing through P. We may assume $P = (p_1, p_2)$ with p_1 on P_1 and p_2 on P_2. According to Theorem 5 there exist lines p^*_1 and p^*_2 with $H_\mathfrak{H}(p^*_1) = p_1$, $H_\mathfrak{H}(p^*_2) = p_2$. Hence the point $P^* = (p^*_1, p^*_2)$ is transformed by $H_\mathfrak{H}$ into P.

Thus the point transformation $H_\mathfrak{H}$ is proved to be $(1, 1)$.

Definition 5. An improper point is called the pole of a line l, if that point is determined by the pencil of perpendiculars to l.

THEOREM 10. *Let P be any improper point, which is not the pole of a*

[10] See Corollary to Theorem 5.

line which passes through the proper point O. Then there exists an ordinary $H_\mathfrak{d}$ *with the center O which transforms P into a proper point.*

Proof. Let $< O, P > = p$, $P = (p, p')$. We denote the line $\perp p'$ and passing through O by n. Then $H_{p.n}(P) = (p' n)$.

THEOREM 11. *Let* $H_\mathfrak{d}$ *and* $H_{\mathfrak{d}'}$ *be two ordinary halfrotations with the same center O. Then* $H_{\mathfrak{d}'^{-1}} \cdot H_\mathfrak{d}(P) = H_\mathfrak{d} \cdot H_{\mathfrak{d}'^{-1}}(P)$ *for any point P.*

Proof. See AG, Theorem 63 and NG, p. 466.

As an application to the theorems concerning ordinary halfrotations, we prove the following theorem of which we shall make use later on.

THEOREM 12. *Let a, b, c be three lines all passing through the same proper point O and let the lines a and b be not orthogonal to c. Then there exists a line l which cuts a and b in two distinct proper points and* $l \perp c$.

Proofs. Let B be any proper point on b, $B \neq O$. We denote the line through B and $\perp c$ by n and we denote (n, c) by C. Whether $(a, n) = A$ is proper or improper, A is certainly not the pole of any line passing through O. Hence there exists an ordinary halfrotation $H_\mathfrak{d}$ which transforms the point A into a proper point and the line n into a line l'. By applying the halfrotation $H_\mathfrak{d}$ to l' we get the line l as required.

If we denote $H_\mathfrak{d}(A)$ by A', $H_\mathfrak{d}(B)$ by B', and $H_\mathfrak{d}(C)$ by C', then we have the relation $H_{a.c}(A') = C' = H_{b.c}(B')$. This fact will also be used later in Theorem 38.

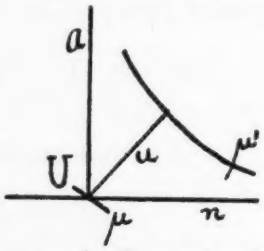
II. Improper Lines; the *O*-lines.

Definition 6. The totality of the points in the plane, which satisfy the following condition will be denoted by $< a; U >$ where U denotes a point lying on a. A point X belongs to $< a; U >$ if, and only if, for any two distinct lines x and y passing through X, the rotation $x \cdot y$ can be represented as the product transformation of the reflexion a and a sliding-reflexion whose axis u passes through U. In symbols the condition may be stated as follows: for $x \neq y$, $X = (x, y)$, $x \cdot y = a \cdot u \cdot \mu \cdot \mu'$ with $\mu \perp u$, $\mu' \perp u$ and u passing through U.

The point U is contained in $< a; U >$ by putting $\mu = \mu'$, and the pole of a is also contained in $< a; U >$ as is easily verified by setting in the formula $u = a$.

THEOREM 13. *Let U be a proper point on a. Then $< a; U >$ is the line orthogonal to a at the point U.*[11]

Proof. Let n be the line $\perp a$ at U. If X is a point on n and $X = (x, y)$, then $x \cdot y = n \cdot \mu'$, where μ' denotes a line on X. u is the line which passes through U and is orthogonal to μ' and $\mu \perp u$ at U. Then we have $n \cdot \mu' = n \cdot a \cdot a \cdot \mu' = a \cdot (n \cdot a)\mu' = a \cdot u \cdot \mu \cdot \mu'$. Hence X belongs to $< a; U >$.



Conversely let $x \cdot y = a \cdot u \cdot \mu \cdot \mu'$ with u passing through U and $\mu \perp u$, $\mu' \perp u$. We can assume that μ passes through U. Then $a \cdot (u \cdot \mu)\mu' = a \cdot (n \cdot a) \cdot \mu' = n \cdot \mu'$. Hence (x, y) lies on n.

THEOREM 14. *If U is an improper point, then all the points on $< a; U >$ are improper.*

Proof. We prove that if a proper point P is contained in $< a; U >$, then $< a; U >$ is the line passing through P orthogonal to a at U. Hence U is proper.

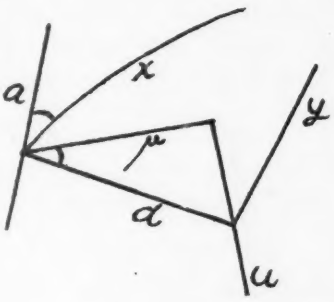
Let p be the line which passes through P and is orthogonal to a. Then we may put $P = (p, y)$. Since P lies on $< a; U >$, we have $p \cdot y = a \cdot u \cdot \mu \cdot \mu'$, with u on U and $\mu \perp u$, $\mu' \perp u$. From $a \cdot p \cdot y = u \cdot \mu \cdot \mu'$ we see at once that u passes through (a, p). Hence $U = (a, p)$.

THEOREM 15. *Let there be a configuration of six lines x, y, a, d, u and μ with the properties: 1°. The four lines x, a, d and μ are concurrent at a proper point O with $x \cdot d = a \cdot \mu$. 2°. The line $y \perp d$ and (d, y) is distinct from O. 3°. The line u passes through (d, y) and is orthogonal to μ. Any halfrotation or inverse-halfrotation with O as center will transform such a configuration into a configuration with the same properties; by this we mean that, on denoting the images of the six lines by x', y' b, d', z and λ, respectively, we have: 1°. The four lines x', b, d' and λ are concurrent at O with $x' \cdot d' = b \cdot \lambda$. 2°. $y' \perp d'$ and $(d', y') \neq O$. 3°. $z \perp \lambda$ and λ passes through (d', y'). For the inverse-halfrotation it must be assumed that the image y' of y really exists.*

Proof. This follows immediately from Theorems 8 and 9. That the

[11] By this we mean that $< a; U >$ is the totality of points lying on the line $\perp a$ at U.

images of x, a, μ and d in an inverse halfrotation really exist, follows from the fact that a halfrotation with the proper point O as center induces a $(1, 1)$ line-transformation in the pencil of lines at O. If y' exists, then (d', y') and, hence, z exist. Finally it may be mentioned that in the configuration the line x cannot be orthogonal to a, for otherwise v would coincide with d.



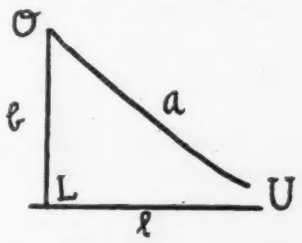
COROLLARY. *Let μ' denote the line $\perp u$ at (d, y) and let λ' be the line $\perp z$ at (d', y'). The two configurations of six lines may be interpreted as $x \cdot y = a \cdot u \cdot \mu \cdot \mu'$ and $x' \cdot y' = b \cdot z \cdot \lambda \cdot \lambda'$. Our theorem states, that $x \cdot y = a \cdot u \cdot \mu \cdot \mu'$ implies $x' \cdot y' = b \cdot z \cdot \lambda \cdot \lambda'$ where x', y', b, z and λ are the images of x, y, a, u and μ, respectively, in a halfrotation or an inverse halfrotation.*[12]

THEOREM 16. *The ordinary halfrotation $H_{a.b}$ transforms $< a; U >$ into $< b; Z >$ if $H_{a.b}(U) = Z$. The inverse halfrotation $H_{a.b}^{-1}$ with the proper point (a, b) as center transforms $< b; U >$ into $< a; Z >$ if $H_{a.b}^{-1}(U) = Z$.*

This follows immediately from the corollary to the above theorem and from the remark that, with $H_{a.b}(a) = b$, the pole of a is transformed into the pole of b by $H_{a.b}$.

THEOREM 17. *Let U be a proper or an improper point determined by two different lines a and l and let O be any proper point, distinct from U, on a. Let b be the line passing through O and orthogonal to l. Then the halfrotation $H_{a.b}$ transforms $< a; U >$ into the line l, if U is not the pole of any line on O.*

Proof. The assumption that U is not the pole of b, means that a and b are not orthogonal. Hence $H_{a.b}$ exists. Since $H_{a.b}(U) = (b, l) \equiv L$, we have $< a; U > \rightarrow < b; L >$ by $H_{a.b}$. By Theorem 13 $< b; L >$ is exactly the line l.



Remark. In case U is proper, this theorem follows from Theorem 8.

THEOREM 18. *Consider an ordinary halfrotation H_δ with the center O.*

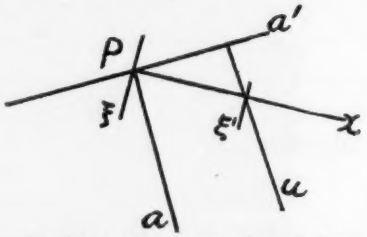
[12] The line λ' is not necessarily the image of μ'. This is true only when the plane is singular-absolute. See V. Theorem 52.

Any totality of points in the plane which can be transformed by $H_{\mathfrak{d}}$ into a given line l can be represented by $< a; U >$, where a is a line passing through O and U is not the pole of any line on O.

Proof. Let b be the line passing through O and $\perp l$, $a = \mathfrak{d} \cdot b$ and $U = (a, l)$. Then $H_{a,b}(U) = (b, l)$. Since $\mathfrak{d}^2 \neq 1$ a is not orthogonal to b and U is not the pole of b.

Definition 7. The polar p of a proper point P is defined as the totality of the poles of the lines which pass through P.

THEOREM 19. *Let a and a' be any two mutually orthogonal lines at the proper point P and let U be the pole of a'. Then the polar p of P is $< a; U >$.*



Proof. p contains the pole U of a' as well as the pole of a since p is the polar of P.

Let now x be any line which passes through P, $x \neq a$, $x \neq a'$ and $\xi \perp x$ at P, $\xi' \perp x$, $\xi' \neq \xi$. Let u be the line which passes through (x, ξ') and is $\perp$ to a'. Then $\xi \cdot \xi' = (\xi \cdot x) \cdot (x \cdot \xi') = a \cdot a' \cdot u \cdot \mu$ with $\mu \perp u$ at (x, ξ'). Since $a' \perp u$, we have $(\xi, \xi') =$ the pole of x which is contained in $< a; U >$.

Conversely, let (ξ, ξ') be any point contained in $< a; U >$. We may assume that ξ passes through P. Then $\xi \cdot \xi' = a \cdot u \cdot \mu \cdot \mu'$ with $u \perp a'$. Hence $\xi \cdot \xi' = a \cdot a' \cdot u \cdot \mu'' \perp u$. From this it follows that ξ and ξ' are the perpendiculars of the line connecting P and (μ'', u). Hence (ξ, ξ') lies on p.

THEOREM 20. *$< a; U >$ is the polar of a proper point if, and only if, U is the pole of a line.*

Proof. If U is the pole of a line, this line must be orthogonal to a. Our theorem follows immediately from Theorem 19.

We are now ready to extend our plane by introducing the so-called "improper O-lines," where O is a proper point arbitrarily chosen in the plane.

Definition 8. What we have up to the present called "lines" will be in future called "proper lines." Let O be a proper point in the plane, then an improper O-line is any $< a; U >$, where a is a proper line which passes through O and U is an improper point on a. Proper and improper O-lines will be also called O-lines. The proper and improper points together with the O-lines form the O-plane.

Lying on an improper O-line there is no proper point. Hence if we know that a line l passes through a proper point, then this line l is certainly proper. We may then occasionally omit the adjective "proper."

If a and b are two distinct lines passing through O, and U (on a) and Z (on b) are both contained in the polar o of O, then $< a; U >$ as well as $< b; Z >$ represent the polar o.

Definition 9. Two O-lines are said to be distinct if the totality of the points lying on the one is not the same as the totality of the points lying on the other.

An improper O-line is always distinct from a proper line, for on an improper line there is no proper point.

THEOREM 21. *Two improper O-lines $< a; U >$ and $< b; Z >$ are not distinct if, and only if,* 1) $a = b$, $U = Z$, 2) *In $< a; U >$ and $< b; Z >$ the points U and Z are contained in o.*

Proof. If one of the two points U and Z, say U, is not contained in o, then $< a; U >$ is distinct from o, and $< a; U > \neq b, Z' = o$. Hence the two O-lines are distinct. If both U and Z are not contained in o, then we have the cases 1, $a = b$; 2, $a \neq b$. In case $a = b$, $U \neq Z$ the two O-lines $< a; U >$ and $< a; Z >$ are distinct because the only point common to a and $< a; U >$ is U, as is immediately seen from the formula $x \cdot y = a \cdot u \cdot \mu \cdot \mu'$. In case $a \neq b$ and $< a; U > = < b; Z >$, the improper O-line $< a; U >$ must contain the pole of a as well as the pole of b. Hence with $\beta \perp b$, $\beta' \perp b$, $\beta' \neq \beta$ we have $\beta \cdot \beta' = a \cdot u \cdot \mu \cdot \mu'$ with $u \neq a$ and u lying on U. But the pole of b is also contained in o, and if $a' \perp a$ at O and W is the pole of a', then $o = < a; W >$ and $W \neq U$. We have then also $\beta \cdot \beta' = a \cdot w \cdot \lambda \cdot \lambda'$ with w on W. Hence $u \cdot \mu \cdot \mu' = w \cdot \lambda \cdot \lambda'$. It follows from Theorem 2 that $u = w$ and $U = W$ which is impossible.

THEOREM 22. *Aside from the polar o of O each O-line has an unique representation as $< a; U >$ where a denotes a line which passes through O and U is a point on a but not contained in o.*

THEOREM 23. *Any improper O-line distinct from o can be transformed by a suitably chosen halfrotation with the center O into a proper line.*

THEOREM 24. *A halfrotation H or an inverse halfrotation H^{-1} with the center O induces a $(1, 1)$ transformation in the O-lines. Together with the H or H^{-1} as a $(1, 1)$ point transformation in the plane, we see that H or H^{-1} is a $(1, 1)$ collineation in the O-plane.*

Proof. H or H^{-1} transforms every point on o into a point on o; hence H or H^{-1} leaves the polar o fixed. Every point U not on o has in H or H^{-1} an unique image Z not on o. The line-transformation induced in the pencil of lines at O is $(1,1)$. From $H(a)=b$ or $H^{-1}(a)=b$ (with a, b on O) and $H(U)=Z$ or $H^{-1}(U)=Z$ we have immediately $<a;U>\rightarrow<b;Z>$ by H or H^{-1}.

THEOREM 25. *In the O-plane the points and the O-lines satisfy the incidence-relations:* 1) *two distinct O-lines have one and only one point in common;* 2) *there is one and only one O-line which passes through two distinct points.*

Proof. With a product of halfrotations with O as center we can always transform two points not on o or two O-lines $\neq o$, respectively, into two proper points and two proper lines. Our theorem follows from the fact, that the halfrotations and their inverses are collineations in the O-plane. The cases where one of the given points is on o or the polar o is one of the given O-lines, require a special discussion which can be furnished without any difficulty.

Remark. For every proper point P there is an extended P-plane obtained by introducing the improper P-lines. We do not know so far whether for any two distinct proper points O and P the extended P-plane is the same as the extended O-plane. We shall prove this fact as in AG by applying Desargues' theorem.

THEOREM 26. *Let P and O be two distinct proper points. Then the polar p of P is always an improper O-line.*

Proof. We have $O \neq P$; let $a=<O;P>$, n be the line $\perp a$ at P. Then we have $p=<a;N>$, where N denotes the pole of n.

III. Theorems of Desargues and Pascal.

THEOREM 27. *Let P be any proper point in the plane whose polar is p. All the Desargues' configurations with the point P as perspective center and the polar p as perspective axis are true in the extended P-plane.*

Proof. We shall prove, in the extended P-plane, the existence of all the perspective collineations with P as center, p as axis.

Let a be any proper line on P. A and A' are two points arbitrarily given

on a with the condition that $A \neq P$ and $A' \neq P$ do not both lie on p. We shall construct the perspective collineation which leaves P and p fixed and transforms A into A'. As is well known, such a collineation, if it exists, will be uniquely determined by A and A'.

Let $b \perp a$ at P and let B be a proper point on b, $P \neq B$. Then the lines $d = < B, A >$ and $d' = < B, A' >$ are both proper and not orthogonal to b nor to a. Let $n \perp d$ and n through P, $n' \perp d'$ and n' also through P. Then n and n' are both distinct from a and b and are not orthogonal to them. Hence we can form the ordinary halfrotations $H_{a,n} \cdot H_{b,n}$ and their inverses $H_{a,n}^{-1}$ and $H_{b,n'}^{-1}$. The product transformation $K = H_{a,n} \cdot H_{b,n}^{-1} \cdot H_{b,n'} \cdot H_{a,n'}^{-1}$ is a collineation in the P-plane, which leaves P and p fixed. Since $K(a) = a$, K transforms every line on P into itself. But $H_{a,n} \cdot H_{b,n}^{-1}(A) = B$, $H_{b,n'} \cdot H_{a,n'}^{-1}(B) = A'$; hence $K(A) = A'$. Then K is the required collineation.

From this theorem follows a special form of Desargues' configuration, which besides P, p and the points on p consists only of proper points and proper lines. These special cases may be stated as follows:

THEOREM 28. *Let P be any proper point in the plane, and let r, s, t be three different proper lines passing through P. Let R_1 and R_2, S_1 and S_2, P_1 and T_2 be proper points lying on r, s and t, respectively, and all distinct from P. If now*

$$< R_1, S_1 > \perp \tau, \qquad < R_2, S_2 > \perp \tau;$$
$$< S_1, T_1 > \perp \rho, \qquad < S_2, T_2 > \perp \rho;$$
$$< T_1, R_1 > \perp \sigma,$$

where ρ, τ, σ are three proper lines passing through P, then $< T_2, R_2 > \perp \sigma$.

We notice that in the above statement no mention is made of the improper P-lines. Therefore this theorem can be regarded as a theorem in the original plane.

We now take a proper point O in the plane and extend the latter to the O-plane. We proceed to prove the validity of the general Desargues' theorem in the O-plane, i. e., Desargues' configurations which consist of points and O-lines.

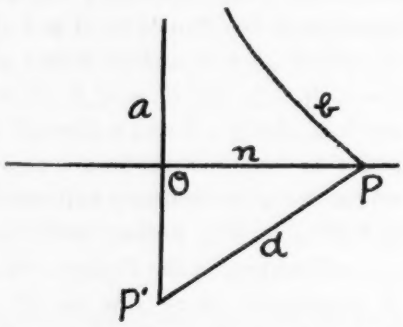
THEOREM 29. *Let P' be any proper point and r', s' and t' be any three distinct lines passing through P'. Let R'_1 and R'_2, S'_1 and S'_2, T'_1 and T'_2 be proper points lying on r', s' and t', respectively, and all distinct from P'. If now*

$$< R'_1, S'_1 > \perp \tau', \qquad < R'_2, S'_2 > \perp \tau';$$
$$< S'_1, T'_1 > \perp \rho', \qquad < S'_2, T'_2 > \perp \rho';$$
$$< T'_1, R'_1 > \perp \sigma',$$

where σ', ρ', τ' are three proper lines which pass through the proper point O, then $< T'_1, R'_2 > \perp \sigma'$.

Proof. If $P' = O$, then this theorem follows from the previous one by putting $P \equiv O$.

We may assume $P' = O$. We denote $< OP' >$ by a and the line $\perp a$ at O by n. We take a proper point $P \neq O$ on n and denote d by $< P, P' >$ and the line $\perp d$ at P by b. Certainly $a \neq b$ and $(a \cdot b)^2 \neq 1$. We form the halfrotation $H_{a.b}$ which may be ordinary or not.



We shall prove that $H_{a.b}$ transforms the configuration stated in this theorem into a configuration which is described in Theorem 28.

According to Theorem 5 the halfrotation $H_{a.b}$ transforms $P' \to P$, $r' \to r$, $s' \to s$, $t' \to t$ and $R'_1 \to R_1$, $R'_2 \to R_2$, $S'_1 \to S_1$, $S'_2 \to S_2$, $T'_1 \to T_1$, $T'_2 \to T_2$ where r, s and t are three lines passing through P and R_1 and R_2, S_1 and S_2, T_1 and T_2 are six proper points distinct from P and lying on r, s, and t, respectively.

We now consider instead of $H_{a.b}$ the halfrotation $H_{b.a}$. We have $H_{b.a}(P) = O$. Hence, again according to Theorem 5, $H_{b.a}$ transforms the lines at P $(1, 1)$ into the lines at O. Therefore there exist three lines σ, ρ and τ which pass through P so that $H_{b.a}(\rho) = \rho'$, $H_{b.a}(\sigma) = \sigma'$ and $H_{b.a}(\tau) = \tau'$.

By our lemma (Theorem 7) we see, incidentally, that the perpendiculars of σ', ρ' and τ' will be transformed into the perpendiculars of σ, ρ and τ respectively. Hence we have:

$$< R_1, S_1 > \perp \tau, \qquad < R_2, S_2 > \perp \tau;$$
$$< S_1, T_1 > \perp \rho, \qquad < S_2, T_2 > \perp \rho;$$
$$< T_1, R_1 > \perp \sigma.$$

According to Theorem 28 $< T_2, R_2 > \perp \sigma$ must hold. As $< T_2, R_2 >$ is the image of $< T'_2, R'_2 >$ in $H_{a.b}$, it follows from the lemma that $< T'_2, R'_2 > \perp \sigma'$.

The above theorem can be stated as a theorem in the O-plane as follows. Any Desargues' configuration in the O-plane, which has a proper point as perspective center and the polar of O as perspective axis and which consists, besides the three improper points on o, of proper points only, is true.

THEOREM 30. *All the Desargues' configurations in the O-plane which contain the polar o of O, are true.*

Proof.[13] First let the perspective center of the configuration be outside o. As such a configuration contains only a finite number of improper points which do not lie on o, we can transform them into proper points by a product of halfrotations. The original configuration will be transformed into a configuration described in Theorem 29. Hence the latter configuration is true. By the inverse process of halfrotations we see, incidentally, the truth of the general configuration.

If the center of the configuration lies on o its validity is shown by an indirect proof.

THEOREM 31. *Desargues' theorem is true in the O-plane.*

THEOREM 32. *Pascal's theorem is true in the O-plane.*

Proof. We prove first a special case of Pascal's theorem. As constructed in Theorem 27 every perspective collineation with center O and axis o is a product of halfrotations and inverse-halfrotations with O as center. Since the halfrotations with the same proper point as center are commutative it follows at once that the perspective collineations with the center O are commutative. From this follows immediately a special case of Pascal's theorem.

The general case will be proved by using the translations in the O-plane.

Having established the validity of Desargues' theorem in the O-plane we can use it to prove that, for any two different proper points O and O', the totality of the points on an improper O-line is the same as the totality of the points on an improper O'-line. Hence the extension of the plane obtained by introducing improper lines does not depend upon the particular proper point O chosen as center.

A proper O-line is also a proper O'-line. The polar of a proper point is an improper O-line as well as an improper O'-line. Given any improper O-line s, which is not a polar, we shall prove the existence of Desargues' configurations which contain besides s only proper lines and polars. As the Desargues theorem holds in the O-plane as well as in the O'-plane, it follows readily that s is also an improper O'-line.

Given an improper O-line s, which is not a polar of any proper point, let o denote the polar of O and let $(o \cdot s) = S$. Let A_1 and A_2 be any two different points on s, each distinct from S. We write $a_1 = < O, A_1 >$, $a_2 = < O, A_2 >$. According to Theorem 12 there exists a proper line s' which cuts a_1 and a_2 in two proper points A'_1 and A'_2 and which passes through S. Let a_3 be another proper line on O, distinct from a_1 and a_2, and let A_3 be a

[13] The proof follows NG, p. 471.

proper point on a_3, $A_3 \neq O$; let $< A_3, A_1 > = r$, $< A_3, A_3 > = t$; then r and t are proper lines. We connect A'_1 and the point (r, o) with the line r', A'_2 and (t, o) with t'. Then we have a Desargues configuration which consists of 8 proper lines a_1, a_2, a_3, t, r, s', r', t' and the polar o and the improper O-line s. From this it follows that A_1, A_2 and S also lie on the same improper O'-line. If we vary the point A_2 and keep A_1 and S fixed, we see at once that s is an improper O'-line.

THEOREM 33. *The extension of the plane obtained by introducing improper lines does not depend upon the particular proper point O chosen as center.*

We shall, in the following, use the terms "improper lines" instead of "improper O-lines" and "lines" instead of "O-lines," the "extended plane" instead of "O-plane."

THEOREM 34. *Let P be any proper point in the plane. Any line in the extended plane can be represented as $< a; U >$ with a passing through P.*

Proof. We can regard the extended plane as the P-plane.

THEOREM 35. *Any motion in the plane induces a collineation in the extended plane.*

Proof. A motion $\mathfrak{M}$ is always a $(1, 1)$ point transformation in the extended plane, because $a \cdot b \cdot c = d$ implies $\mathfrak{M}(a) \cdot \mathfrak{M}(b) \cdot \mathfrak{M}(c) = \mathfrak{M}(d)$. If, now, an improper line is represented by $< a; U >$ and $\mathfrak{M}(a) = a'$, $\mathfrak{M}(U) = U'$, then $\mathfrak{M}$ transforms $< a; U >$ into $< a'; U' >$, since relations of the form $\chi \cdot y = a \cdot u \cdot \mu \cdot \mu'$ must also hold among the image lines.

IV. Polarity. Quadratic Form.

Definition 10. If there exists a rectangle in the plane, the geometry is called singular absolute, otherwise ordinary absolute.[14]

THEOREM 36. *All the proper points in a singular absolute geometry have the same polar—the absolute polar.*

Proof. By assumption there exists a rectangle formed, say, by the four proper lines γ, γ', δ and δ' with the $\gamma, \gamma' \perp \delta, \delta'$. Then if γ'' is a third proper line $\gamma'' \perp \delta$, the product $\gamma \cdot \gamma' \cdot \gamma''$ is a reflexion (AG, Theorem 20). Hence, according to Theorem 21 in AG, γ'' is also orthogonal to δ' as χ and γ' are.

[14] According to the definition given by Reidemeister and Bachmann.

Thus the two proper lines γ and γ' have the same pencil of perpendiculars, which we denote by $\mathfrak{P}_1$. Since the same holds true for δ and δ', we may denote the pencil of their common perpendiculars by $\mathfrak{P}_2$. Thus we have an orthogonal net in the plane. Through every proper point there are two mutually orthogonal lines, the one belonging to $\mathfrak{P}_1$, the other to $\mathfrak{P}_2$. All the proper lines in $\mathfrak{P}_1$ have the same pole P_1 and all the proper lines in $\mathfrak{P}_2$ have the same pole P_2.

Thus the polar of any proper point passes through P_1 and P_2. Since $P_1 \neq P_2$, this polar is uniquely determined by P_1 and P_2. Hence all the proper points have the same polar.[15]

THEOREM 37. *In an ordinary absolute geometry different proper points have always different polars.*

THEOREM 38. *To every improper point U, which is not the pole of a proper line, there exists uniquely an improper line ω—called the polar of the improper point U*[16]*—with the property, that any proper line, which passes through U, has its pole lying on ω.*

Proof. We take a proper line a which passes through U, and on this line take a proper point O. We can form the improper line $< a; U >$ which is not the polar of any proper point (Theorem 22).

Let $u_1 \neq a$ be any proper line passing through U and $\beta_1 \perp u_1$, β_1 on O. Since U is not the pole of a proper line $(a \cdot \beta_1)^2 \neq 1$ and $H_{\beta_1 . a}$ exists. This halfrotation $H_{\beta_1 . a}$ will transform the pole of u_1 into a point, say Z_1. Let β'_1 be another perpendicular of u_1, $\beta'_1 \neq \beta_1$, and let $H_{\beta_1 . a}(\beta'_1) = z_1$; then we have evidently $Z_1 = (z_1, a)$ since $H_{\beta_1 . a}(\beta_1) = a$. According to our lemma the halfrotation $H_{a . \beta_1}$, will transform the pole of z_1 into the pole of β'_1. But the pole of β'_1 is a point on u_1 and so the pole of z_1 is a point belonging to $< a; U >$.

If we take another proper line u_2 which passes through U and $u_2 \neq a$, $u_2 \neq u_1$ and let β_2 be the perpendicular of u_2 through O, then the halfrotation $H_{\beta_2 . a}$ will transform the pole of u_2 into a point, say, Z_2. And if β'_2 is another perpendicular of u_2, $\beta'_2 \neq \beta_2$ and $H_{\beta_2 . a}(\beta'_2) = z_2$ then evidently we have $Z_2 = (z_2, a)$. And as proved above for z_1, the proper line z_2 has its pole lying on $< a; U >$.

According to Theorem 12 there exist two proper points P_1 and P_2 with P_1 not on β_1 and P_2 not on β_2, so that $H_{\beta_1 . a}(P_1) = P = H_{\beta_2 . a}(P)$. We may assume that β'_1 passes through P_1 and that β'_2 through P_2. Then z_1 as well

[15] Another simple proof of this theorem, which does not make use of the extended plane, is given in the Appendix.

[16] Up to the present we have only defined the polar of a proper point (Definition 7).

as z_2 must pass through the same proper point P. If $z_1 \neq z_2$, then $< a; U >$, which contains the poles of z_1 and z_2, would be the polar of the proper point P. This is impossible. Hence $z_1 = z_2$ and $(z_1, a) = Z_1 = Z_2 = (z_2, a)$.

It follows that whatever the proper line u_i may be, provided u_i passes through U, the halfrotation $H_{\beta_i . a}(\beta_i \perp u_i \beta_i$ on $O)$ always transforms the pole of u_i into the same point, which we denote by Z. Or in symbols we have

$$\beta'_i \cdot (\beta_i \cdot a) = z_i \lambda \lambda' \quad \text{with} \quad \lambda \perp z_i,\ \lambda' \perp z_i,\ z_i \quad \text{on} \quad Z.$$
$$\beta_i \cdot \beta'_i = a \cdot z_i \cdot \lambda' \cdot \lambda.$$

In words: the pole (β_i, β'_i) of u_i lies on $< a; Z >$. If we put $\omega = < a; Z >$ then ω is the line required.

If there were two distinct lines ω, ω' with the same property, then all the proper lines, which pass through U would have the same point (ω, ω') as pole. Then U would be the pole of any proper line orthogonal to the u_i. This contradicts our assumption.

THEOREM 39. *To every improper line ω, which is not the polar of a proper point, there exists uniquely an improper point Z—called the pole of the improper line ω—with the property, that the proper lines, which pass through Z, and only these, have their poles lying on ω.*

Proof. We denote ω by $< a; U >$, where U is an improper point, which is not the pole of any proper line. Starting from U we construct, as in Theorem 38, the point $Z = (z_i, a)$, so that the pole of z_i lies on $< a; U > = \omega$. Since the pole of a is certainly contained in $< a; U >$, we have at least two different proper lines a and z_i which pass through Z and have their poles lying on ω.

Z must be improper for otherwise ω would be the polar of a proper point. Hence either Z is the pole of a proper line or Z is not the pole of any proper line. In the latter case we apply Theorem 38 to the improper point Z, from which it follows, that all the poles of the proper lines passing through Z must lie on the same line. Hence this line must be ω. Therefore Z can be defined as the pole of ω.

In case Z is the pole of a proper line l, then the poles of a and z_i must be the same point (ω, l). Hence all the proper lines on Z have this point as their pole.

Any proper line m which does not pass through Z, can not have its pole lying on ω, for m must intersect some proper lines on Z in proper points.

By applying the above two theorems in the singular absolute and ordinary absolute geometries we shall prove the existence of an involution and a polarity.

THEOREM 40. *Let the geometry be singular absolute. The polar of an improper point, which is not the pole of a proper line, is the absolute polar* π. *The pole of an improper line* ω, *which is not the absolute polar, is the same as the pole of proper lines which pass through* (ω, π). *Hence follows the existence of an involution, a* $(1, 1)$ *point transformation on* π *induced by the right angles at any proper point. If in this involution* A *and* A' *are a pair of corresponding points, then* $A \neq A'$ *and the lines passing through* A *have* A' *as pole and those passing through* A' *have* A *as pole.*

Proof. For the proof it is sufficient to remark, that any two different proper lines p and p' may have the same pole or different poles according as the point (p, p') is itself a pole or not. In the latter case the polar of (p, p') must be the absolute polar.

THEOREM 41. *Let the geometry be ordinary absolute. To every point (proper or improper!) there exists a line called its polar and to every line (proper or improper!) there exists a point called its pole. Different points have different polars, different lines have different poles. If* P *has* p *as polar, then* p *has* P *as pole.*

Proof. For the proof it is sufficient to remark, that different proper lines have always different poles.

THEOREM 42. *The polarity in an ordinary absolute geometry is a correlation and a reciprocity.*

This theorem is equivalent to

THEOREM 43. *If a point* P *lies on a line* l, *then the polar* p *of* P *passes through* L, *the pole of* l.

In order to prove this theorem, we need the following generalization of our lemma:

THEOREM 44. *Let* $H_{\mathfrak{H}}$ *be an ordinary halfrotation. If* $H_{\mathfrak{H}}(P) = P'$, *then* $H_{\mathfrak{H}^{-1}}$ *transforms the polar* π' *of* P' *into the polar* π *of* P. *Conversely if* $H_{\mathfrak{H}}$ *transforms a line* π *into a line* π', *then* $H_{\mathfrak{H}^{-1}}$ *transforms the pole* P' *of* π' *into the pole* P *of* π.

Proof. From $H_{\mathfrak{H}}(P) = P'$, it follows that every proper line p passing through P will be transformed into a proper line p', which passes through P'. Hence, according to our lemma, $H_{\mathfrak{H}^{-1}}$ transforms the pole of p' into the pole of p. Since $H_{\mathfrak{H}^{-1}}$ is a collineation on the extended plane, and $H_{\mathfrak{H}^{-1}}$ transforms more than one point on π' into points on π, so $H_{\mathfrak{H}^{-1}}(\pi') = \pi$.

Conversely if $H_{\mathfrak{H}}(\pi) = \pi'$, then $H_{\mathfrak{H}^{-1}}(P') = P$. For otherwise from $H_{\mathfrak{H}^{-1}}(P') = P''$ it would follow that $H_{\mathfrak{H}}(\pi'') = \pi'$. Hence $\pi'' = \pi$ and $P'' = P$.

Proof of Theorem 43. Let P be a point on l, and let l be a proper line. The polar π of P is defined as the line containing all the poles of proper lines passing through P. Since l is such a proper line, we see at once that the pole L of l must be contained in π; in other words π passes through L.

Let now l be an improper line. Then there always exists an ordinary halfrotation $H_{\mathfrak{H}}$ which transforms l into a proper line l'. $H_{\mathfrak{H}}(l) = l'$, $H_{\mathfrak{H}}(P) = P'$. P' lies on l'. According to the part of the theorem just proved, the polar π' of P' must pass through the pole L' of l', since l' is proper. From Theorem 44 we have $H_{\mathfrak{H}^{-1}}(\pi') = \pi$ and $H_{\mathfrak{H}^{-1}}(L') = L$. Hence π passes through L, because $H_{\mathfrak{H}^{-1}}$ is a collineation in the extended plane.

THEOREM 45. *Any motion in the plane leaves the involution on the absolute polar or the polarity in the ordinary absolute geometry invariant.*

Proof. Any motion is a collineation in the extended plane. Furthermore any motion leaves the orthogonality-relations invariant. This proves the theorem.

We now proceed to show that the polarity is not only a correlation, it is also a *projective correlation*. From this follows immediately the existence of a quadratic form.

A collineation or a correlation K is certainly projective, if it transforms a certain one-dimensional form F_1 into the same or another one-dimensional form F_2, in such a way that the mapping induced by K in the elements of F_1 and F_2 is projective, that is to say, if the mapping leaves the cross-ratios unchanged. We shall use this remark to prove the existence of a quadratic form as follows:

1°. Any reflexion in the plane is a projective transformation in the extended plane.

Proof. A reflexion leaves every point on its axis fixed; it is furthermore a collineation in the plane. Hence it is a projective transformation of period 2. From this it follows, also, that the coördinate field of our geometry is not of characteristic 2.

2°. Any rotation in the plane is a projective transformation.

3°. Any ordinary halfrotation is a projective transformation.

Proof. Suppose that the ordinary halfrotation $H_{\mathfrak{d}}$ has the proper point O as center. The polar o of O will be transformed into itself by $H_{\mathfrak{d}}$ and $\mathfrak{d}$. We introduce homogeneous coördinates $[x_1 : x_2 : x_3]$ in the plane, with the line o as $x_3 = 0$. If a proper point $P[\xi : \eta : 1]$ is transformed into the proper point $P'[\xi' : \eta' : 1]$ by $\mathfrak{d}$ and into the proper point $\bar{P}[\bar{\xi} : \bar{\eta} : 1]$ by $H_{\mathfrak{d}}$ then $\bar{P}$ is the mid-point of P and P' and $< O, \bar{P} > \perp < P, P' >$. Hence the reflexion which has the proper line $< \bar{P}, O >$ as axis transforms P into P' and the points $\bar{P}$ and the point of intersection P_0 of $< P, P' >$ with o are fixed. Therefore $PP'; \bar{P}P_0$ form a harmonic point set. As P_0 lies on o, it must have the coördinate $[\xi - \xi' : \eta - \eta' : 0]$. Hence it follows that the coördinate of $\bar{P}$ is $[\frac{1}{2}(\xi + \xi') : \frac{1}{2}(\eta + \eta') : 1]$.

We represent the rotation $\mathfrak{d}$ in the affine coördinates (x_1, x_2) with o as the line of infinity) by a linear transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = (A) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1}$$

where (A) denotes a square matrix. From this equation we get the equations of transformation $H_{\mathfrak{d}}$ so far as the proper points (x_1, x_2) are concerned:

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \tfrac{1}{2}[(E) + (A)] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2}$$

where (E) is the unit matrix. The equation (2) represents a linear transformation $\mathfrak{T}$ in the extended affine plane. So far as the proper points are concerned, the transformation $\mathfrak{T}$ and $H_{\mathfrak{d}}$ are the same. Hence $\mathfrak{T} \cdot H_{\mathfrak{d}}^{-1}$ leaves every proper point fixed. But if every proper point is a fixed point, then all the points in the extended plane are unchanged by the transformation $\mathfrak{T} H_{\mathfrak{d}}^{-1}$. Hence $H_{\mathfrak{d}} = \mathfrak{T}$, and $H_{\mathfrak{d}}$ is represented as a linear transformation (2).

THEOREM 46. *The involution of right angles at a proper point is a projective transformation.*

Proof. The involution of right angles at a proper point P is induced by the product of halfrotations $H_{a,n} \cdot H_{b,n}^{-1}$ as described in Theorem 27. Since halfrotations are linear transformations the involution is projective.

THEOREM 47. *In the singular absolute geometry there exists a quadratic form $x_1^2 + kx_2^2$ which will be left invariant by those motions in the plane which are products of reflexions. $(-k)$ is not equal to a square in the coördinate field.*

THEOREM 48. *The polarity in the ordinary absolute geometry is a linear reciprocity. Hence there exists a quadratic form $x_1^2 + k_2 x_2^2 + k_3 x_3^2$ with*

$k_2 \cdot k_3 \neq 0$ *which is left invariant by those motions*[17] *in the plane, which are products of reflexions.*

Definition 11. A singular absolute geometry is called euclidean, if there exist no improper points other than those on the absolute polar. When such points exist we call the geometry semi-euclidean.

Definition 12. An ordinary absolute geometry is said to be of elliptic type, if there is no point which lies on its polar. The geometry is called elliptic, if every improper point is the pole of a proper line; otherwise it is termed semi-elliptic.

Definition 13. An ordinary absolute geometry is said to be of hyperbolic type, if there exists at least one point which lies on its polar. Such points are called boundary points. The geometry is called hyperbolic, if, save for boundary points, every improper point is the pole of a proper line; otherwise the geometry is called semi-hyperbolic.

The elliptic geometry has been studied by E. Podehl and K. Reidemeister in: "Eine Begründung der elliptischen Geometrie," *Hamburger Abhandlungen*, vol. 10 (1934). It is easy to see that our method of algebraization can be applied there equally well. There is no need to introduce improper lines and things are simpler.

V. Hyperbolic Geometry.

Hyperbolic geometry will be obtained by directly introducing to the axioms given in AG, § 1 some new axioms, which specify the nature of improper points. We assume that improper points have been introduced into the absolute geometry according to AG, § 3. We shall have nothing to do with the improper lines before we define them, which can be done much more explicitly in this special case. The results developed above concerning the polarity will not be used.

We first state some theorems which can be proved easily by using the axioms of absolute geometry.

THEOREM 49. *A rotation with a proper point O as center leaves no point different from O fixed.*

THEOREM 50. *A sliding-reflexion with the line l as axis will transform a point P, which does not lie on l, into itself if and only if P is the pole of l.*

[17] We do not know whether every motion is a linear collineation, and in the non-euclidean case, whether every linear collineation, which leaves the quadratic form invariant, is a motion.

Proof. Let the sliding-reflexion be $l \cdot \alpha \cdot \alpha'$. If $\alpha = \alpha'$, the theorem is true. Let $\alpha \neq \alpha'$, $(\alpha, l) = A$, $< A, P > = p$, $\beta \perp p$ at A. Then $l \cdot \alpha \cdot \alpha' = p \cdot \beta \cdot \alpha'$. From the assumption $l \cdot \alpha \cdot \alpha'(P) = P$ follows $p \cdot \beta \cdot \alpha'(P) = P$ so that $\beta(P) = \alpha'(P)$; hence $\alpha' \perp p$, which is impossible unless P is the pole of l.

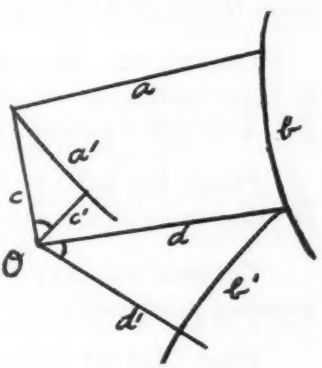
THEOREM 51. *Let α and α' be two different perpendiculars of a line l. The rotation $\alpha \cdot \alpha'$ transforms a point P not lying on l into itself if and only if P is the pole of l.*

Proof. From $\alpha \cdot \alpha'(P) = P$ follows $p \cdot \alpha \cdot \alpha'(P) = P$ if p connects (α, l) with P. But $p \cdot \alpha \cdot \alpha'$ is a sliding-reflexion with the axis passing through (α', l). By applying Theorem 50 we obtain the theorem immediately.

Axiom A. O. There is no rectangle in the plane.

THEOREM 52. *An ordinary halfrotation $H_{\mathfrak{d}}$ with O as center, will not transform two mutual orthogonal lines a, b which both do not pass through O, into mutually orthogonal lines.*

Proof. Let c be the line through O and $\perp a$, and let d be the line through O and $\perp b$. $H_{\mathfrak{d}}(c) = c'$, $H_{\mathfrak{d}}(d) = d'$, $H_{\mathfrak{d}}(a) = a'$, $H_{\mathfrak{d}}(b) = b'$. Then a' passes through (c, a) and b' passes through (b, d).



According to our lemma $H_{\mathfrak{d}^{-1}}$ will transform any line $\perp b'$ into a line $\perp b$. If $a' \perp b'$, then $H_{\mathfrak{d}^{-1}}(a') = a'' \perp b$. But $\mathfrak{d}^{-1} = c' \cdot c$, and a'' passes through (c', a') and $\perp c$. Hence a, b, c and a'' form a rectangle, which contradicts our axiom A. O.

Axiom B_1. There are on a line at most two improper points, which are not the poles of (proper) lines.

Definition 14. Improper points, which are not the poles of lines in the plane, are called, if they exist, boundary points.

THEOREM 53. *If on a line l there is a boundary point R there exist exactly two boundary points on l.*

Proof. Let $\alpha \perp l$. Then $\alpha(R) \neq R$ since R is not the pole of α. But if R is not a pole, $\alpha(R)$ is not a pole either. Hence $\alpha(R)$ is a boundary point. According to Axiom B_1, R and $\alpha(R)$ are the two boundary points on l. The improper points on l distinct from R and $\alpha(R)$ are poles.

Axiom B_2. There exists at least one boundary point in the plane.

Remark. If we take all the lines which pass through a boundary point R, then we get on each line besides R one other boundary point. All these boundary points are distinct. We shall show later, that these are exactly all the boundary points which exist in the plane.

THEOREM 54. *If R is a boundary point lying on the line r, $\alpha \perp r$, $\alpha' \perp r$, then $\alpha \cdot \alpha'(R) = R$.*

THEOREM 55. *If a and r are two lines, which pass through a boundary point R, then a rotation, which can be represented by $a \cdot r \cdot \alpha \cdot \alpha$ with $\alpha \perp r$, $\alpha' \perp r$ will leave the point R fixed.*

THEOREM 56. *Let $x \cdot y = a \cdot r \cdot \alpha \cdot \alpha'$ with a, r passing through a boundary point R; $\alpha, \alpha' \perp r$, $x \perp r'$ where r' is also a line on R, then $y \perp r'$.*

Proof. From a proper point Y on y we take the perpendicular y' of r'. Then $x \cdot y'(R) = R$ (Theorem 54). Hence $y' \cdot y = (y' \cdot x)(a \cdot r \cdot \alpha \cdot \alpha')$ leaves R fixed, and we have $y = y'$ (Theorem 49).

THEOREM 57. *Given a boundary point R and a line a passing through R. Any rotation in the plane which leaves R fixed can be represented as $a \cdot r \cdot \alpha \cdot \alpha'$ with r passing through R and $\alpha \perp r$, $\alpha' \perp r$.*

Proof. Let $\mathfrak{d}(R) = R$, then $a \cdot \mathfrak{d}(R) = R$. But $a \cdot \mathfrak{d}$ is a sliding-reflexion, hence its axis must pass through R (Theorem 50).

THEOREM 58. *Any point, which is contained in $< a; R >$ is either the boundary point itself or the pole of a line passing through R.*

Proof. Let the point P in $< a; R >$ be (x, y) with $x \cdot y = a \cdot r \cdot \alpha \cdot \alpha'$, $\alpha \perp r$, $\alpha' \perp r$, r on R. We can assume $\alpha \neq \alpha'$. To prove that there exists a line r' passing through R with $x \perp r'$, $y \perp r'$, amounts to showing that P is the pole of r'.

1°. P can not be a proper point, for $x \cdot y$ would then not leave R fixed.

2°. P can not be the pole of a line not passing through R according to Theorem 51.

3°. If P is a boundary point distinct from R, then we get a contradiction by the following argument: We take a proper point O on a and let α be $\perp r$ and pass through O. Let d denote the line connecting O and (α', r). We get x and y by the conditions: $x \cdot d = a \cdot \alpha$, $y \perp d$ at (α', r). The ordinary

halfrotation $H_{x.a}$ transforms P into R and hence $< x; P >$ into $< a; R >$. Let d' be another perpendicular of y, $d' \neq d$, then (d, d') belongs to $< x; P >$ and is transformed into a point of $< a; R >$. But $H_{x.a}(d) = \alpha$. Hence according to Theorem 56 $H_{x.a}(d')$ is a line $\perp r$. This is impossible, by Theorem 52, since $H_{x.a}(y) = r$, $\alpha \perp r$, $d \perp y$, $d' \perp y$.

4°. The only possibility remaining for P is that P is the pole of a line passing through R.

THEOREM 59. *All the rotations whose centers are either the boundary point R or the pole of any line passing through R form a group: the group of all rotations in the plane which leave R fixed.*

In proving Theorem 58, 3°, we have used the fact that an ordinary halfrotation can never transform a boundary point into another boundary point. Now it is also easy to prove that an ordinary halfrotation never transforms a boundary point into a pole. If $H_{\mathfrak{d}}$ transforms a boundary point R into the pole of a line l, then $H_{\mathfrak{d}^{-1}}$ will transform l into a line l', which must have R as the pole, according to our lemma. This is impossible. Hence by an ordinary halfrotation a boundary point is always transformed into a proper point. From this follows

THEOREM 60. *Given a boundary point R and a line s not passing through R then there exists a perpendicular of s which passes through R.*

Proof. We take a proper point O on s and let $r = < O, R >$. We assume r not $\perp$ to s. $H_{r.s}$ transforms R into a proper point P. The perpendicular to s at P certainly passes through R.

THEOREM 61. *Any two distinct boundary points are connected by a line.*

Proof. Let R_1 and R_2 be two distinct boundary points, $R_1 = (a, b)$. Then we can write $R_2 = (p, q)$ with $p \perp a$, $q \perp b$ (Theorem 60). Let l be the line connecting (a, p) and (b, q), $t \perp l$ and t passing through R_1. Then the line r with $r = p \cdot t \cdot q$ connects R_2 and R_1 as is easily seen from the relation

$$a \cdot r \cdot b = (a \cdot p) \cdot t \cdot (q \cdot b) = (lt_1)t(t_2 l) = t'$$

with t_1, t_2, t' all $\perp l$.

Definition 15. Improper lines in the hyperbolic geometry. There are two kinds of improper lines, 1) Polar of a proper point, 2) Tangent at a boundary point R.—This consists of R and all the poles of the lines which pass through R.

THEOREM 62. *Two distinct lines (proper or improper) have one and only one point in common. Two distinct points lie on one and only one line (proper or improper).*

Proof. This theorem may be proved by discussing the different cases with regard to the kinds of lines and points which are given. By applying Theorems 61 and 62 and the relations of orthogonality among proper lines the proof follows without any difficulty.

The proofs of Desargues' and Pascal's theorems in the hyperbolic geometry follow the same principle as in III.

The pole-polar relation has been fixed by the definition of improper lines.

Appendix.

We give a simple proof of the theorem: *In case there exists a rectangle in the plane, then for any two different (proper) lines a and b with a common perpendicular, the pencil of lines orthogonal to a is the same as the pencil of lines orthogonal to b.*

Proof. Let the rectangle be formed by the four lines γ, γ', δ, and δ'. Then according to the proof in Theorem 36 we know, that there exists an orthogonal net in the plane consisting of two pencils of lines $\mathfrak{P}_1$ and $\mathfrak{P}_2$.

Now let a and b have a common perpendicular c. We may suppose that a and b are not contained in $\mathfrak{P}_1$ nor in $\mathfrak{P}_2$. Let γ_1 of the pencil $\mathfrak{P}_2$, and δ_1 of the pencil $\mathfrak{P}_1$ both pass through (a, c), $\gamma_1 \perp \delta_1$. We take a proper point P on δ_1, $P \neq (a, c)$. Let $c' \perp c$ and c' pass through P, and let δ_2 be the line of $\mathfrak{P}_1$ which passes through (c', c). Let γ_2 be contained in $\mathfrak{P}_2$, and on P; and let γ_3 be contained in $\mathfrak{P}_2$ and pass through (c', c). Finally, let a' be the line $\perp c'$ at P.

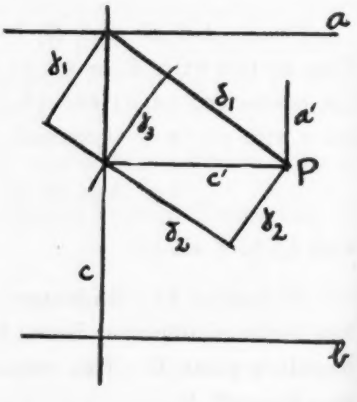
We have

$$c \cdot a = \gamma_1 \cdot \delta_1, \qquad a' \cdot c' = \delta_1 \cdot \gamma_2;$$

hence

$$a \cdot a' = c \cdot \gamma_1 \cdot \gamma_2 \cdot c',$$
$$c \cdot a \cdot a' \cdot c = \gamma_1 \cdot \gamma_2 \cdot c' \cdot c$$
$$= \gamma_1 \cdot \gamma_2 \cdot \gamma_3 \cdot \delta_2 = \gamma_0 \cdot \delta_2$$

with $\gamma_0 \perp \delta_2$. Therefore $a \perp a'$. Thus the four lines a, a', c, c' form a rectangle. From $b \perp c$ follows $b \perp a'$.



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ON LEBESGUE SUMMABILITY AND ITS GENERALIZATION TO INTEGRALS.*

By OTTO SZÁSZ.

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1. Introduction. A series $\sum_1^\infty a_n$ is called Lebesgue summable (shortly L-summable) to the value s, if the sine series

$$\sum_1^\infty n^{-1}a_n \sin nt = F(t) \tag{1.1}$$

is convergent in some interval $-\tau < t < \tau$, and if

$$t^{-1}F(t) \to s \text{ as } t \to 0; \tag{1.2}$$

we then write: $L\Sigma a_n = s$. Let

$$\gamma_n^r = \binom{n+r}{n}, \text{ and } s_n^r = \sum_1^n \gamma_{n-\nu}^r a_\nu; \tag{1.3}$$

the series Σa_n is summable (C, r) for some $r > -1$, to the sum s, if

$$s_n^r \sim s\gamma_n^r \text{ as } n \to \infty; \tag{1.4}$$

we then write $(C, r)\Sigma a_n = s$. Note that if r is not a negative integer, then

$$\gamma_n^r \sim \frac{n^r}{\Gamma(r+1)} \text{ as } n \to \infty. \tag{1.5}$$

Hardy and Littlewood [2, Theorem 2][1] proved that $(C, -\alpha)$ summability for some $\alpha > 0$ implies L-summability. For $\alpha = 0$ the statement no longer holds. In fact Hardy and Littlewood proved [1b, § 2], that to any positive function $\phi(n)$, tending steadily to infinity with n, it is possible to find a convergent series Σa_n, for which $a_n = O\{n^{-1}\phi(n)\}$, while

$$\limsup_{t\to 0} \sum a_n \frac{\sin nt}{nt} = \infty.$$

We replace $(C, -\alpha)$ summability by a more general assumption in the following theorem:

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* Received October 23, 1944. Presented to the American Mathematical Society, October 28, 1944.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

THEOREM A. *If Σa_n is summable $(C, 1-\alpha)$ for some positive $\alpha < 1$, and if*

$$\sigma_n = \sum_1^n | s_\nu{}^{-\alpha} | = O(n^{1-\alpha}) \text{ as } n \to \infty, \tag{1.6}$$

then the series is L-summable.

The proof is contained in **2** and **3**. In the later sections we give several generalizations of L-summability to integrals.

2. An auxiliary theorem. The following formulae are well known:

$$(1-x)^{-r-1} = 1 + \sum_1^\infty \frac{(r+1)(r+2)\cdots(r+n)}{n!}\, x^n = \sum_0^\infty \gamma_n{}^r x^n, \tag{2.1}$$

$$\sum_0^n \gamma_\nu{}^r = \gamma_n{}^{r+1}, \qquad \gamma_n{}^r = \gamma_n{}^{r+1} - \gamma_{n-1}{}^{r+1}, \tag{2.2}$$

$$s_n{}^r = \sum_1^n \gamma_{n-\nu}{}^{\epsilon-1} s_\nu{}^{r-\epsilon}, \qquad a_n = \sum_1^n \gamma_{n-\nu}{}^{-r-2} s_\nu{}^r. \tag{2.3}$$

We first prove the following lemma:

LEMMA 1. *For $\alpha < 2$ the series*

$$\sum_{n=\nu}^\infty \gamma_{n-\nu}{}^{\alpha-2} \frac{\sin nt}{nt} = \sum_{n=0}^\infty \gamma_n{}^{\alpha-2} \frac{\sin(n+\nu)t}{(n+\nu)t} = T_\nu{}^\alpha(t) \tag{2.4}$$

is absolutely convergent, and

$$T_\nu{}^\alpha - T_{\nu+1}{}^\alpha = T_\nu{}^{\alpha-1}, \qquad (\nu = 1, 2, 3, \cdots). \tag{2.5}$$

Furthermore for $\alpha \leq 1$

$$| T_\nu{}^\alpha(t) | < 2\nu^{-1} t^{-\alpha}, \qquad t > 0, \quad \nu \geq 1. \tag{2.6}$$

Proof. If α is an integer ≤ 1, then clearly the series (2.4) is finite; if α is not an integer, then from (1.5)

$$\gamma_n{}^{\alpha-2} \frac{\sin(n+\nu)t}{(n+\nu)t} = O(n^{\alpha-3});$$

hence the series (2.4) is absolutely convergent for $\alpha < 2$. Furthermore, employing (2.2) we find

$$T_\nu{}^\alpha - T_{\nu+1}{}^\alpha = \frac{\sin \nu t}{\nu t} + \sum_{\nu+1}^\infty \gamma_{n-\nu}{}^{\alpha-3} \frac{\sin nt}{nt} = \sum_\nu^\infty \gamma_{n-\nu}{}^{\alpha-3} \frac{\sin nt}{nt};$$

this proves (2.5). To prove (2.6), note that from (2.1)

$$\sum_0^\infty \gamma_n{}^{\alpha-2} x^{n+\nu} = x^\nu (1-x)^{1-\alpha};$$

hence

$$\sum \gamma_n^{\alpha-2} \cos(n+\nu)t = \Re\, e^{i\nu t}(1-e^{it})^{1-\alpha}.$$

On putting $1-\alpha = 2\beta \geq 0$, we obtain

$$e^{i\nu t}(1-e^{it})^{2\beta} = e^{it(\nu+\beta)}(-2i\sin(t/2))^{2\beta} = e^{it(\nu+\beta)}e^{-i\beta\pi}(2\sin(t/2))^{2\beta},$$

and

$$\Re\, e^{i\nu t}(1-e^{it})^{2\beta} = (2\sin(t/2))^{2\beta}\cos\{(\nu+\beta)t-\beta\pi\}.$$

Hence

$$T_\nu^\alpha(t) = t^{-1}\int_0^t (2\sin(x/2))^{2\beta}\cos\{(\nu+\beta)x-\beta\pi\}dx;$$

if $\alpha = 1$, $\beta = 0$, $T_\nu(t) = \sin\nu t/\nu t$, and

$$|T_\nu(t)| < 1/\nu t \text{ for } t > 0, \qquad \alpha = 1, \quad \nu \geq 1.$$

If $\alpha < 1$, $\beta > 0$, an integration by parts yields

$$T_\nu^\alpha(t) = t^{-1}(2\sin(t/2))^{1-\alpha}\frac{\sin\{(\nu+\beta)t-\beta\pi\}}{\nu+\beta}$$

$$-(2\beta/t)\int_0^t (2\sin(x/2))^{-\alpha}\frac{\sin\{(\nu+\beta)x-\beta\pi\}}{\nu+\beta}\cos(x/2)dx;$$

hence

$$|T_\nu^\alpha(t)| < \frac{t^{-\alpha}}{\nu} + \frac{2\beta}{\nu+\beta}\,t^{-1}\int_0^t (2\sin(x/2))^{-\alpha}\cos(x/2)dx$$

$$= \frac{t^{-\alpha}}{\nu} + \frac{1}{\nu+\beta}\,t^{-1}(2\sin(t/2))^{1-\alpha} < 2\nu^{-1}t^{-\alpha}.$$

This proves (2.6), and establishes our lemma.

Remark. A. F. Andersen defines differences of fractional order of a sequence $\{\epsilon_\nu\}$ by the formula

$$\Delta^r\epsilon_\nu = \sum_{n=0}^{\infty}\gamma_n^{-r-1}\epsilon_{\nu+n};$$

hence

$$T_\nu^\alpha(t) = \Delta^{1-\alpha}\,\frac{\sin\nu t}{\nu t}.$$

We have from (2.1), for $r = \alpha - 2$,

$$\gamma_n^{\alpha-2} = (-1)^n\binom{1-\alpha}{n}.$$

3. Proof of Theorem A. Using (2.3)

$$(3.1)\quad \sum_1^\infty n^{-1}a_n\sin nt = \sum_{n=1}^\infty n^{-1}\sin nt\sum_{\nu=1}^n \gamma_{n-\nu}^{\alpha-2}s_\nu^{-\alpha} = \sum_{\nu=1}^\infty s_\nu^{-\alpha}\sum_{n=\nu}^\infty n^{-1}\gamma_{n-\nu}^{\alpha-2}\sin nt;$$

the interchange of summation is legitimate, as the double series is absolutely convergent for $\alpha < 1$. In fact from (1.5)

$$n^{-1}\gamma_{n-\nu}{}^{\alpha-2}s_\nu{}^{-\alpha}\sin nt = O\{n^{-1}(n-\nu+1)^{\alpha-2}\mid s_\nu{}^{-\alpha}\mid\},$$

and

$$\sum_{n=\nu}^{\infty} n^{-1}(n-\nu+1)^{\alpha-2} < \nu^{-1}\sum_{1}^{\infty} k^{\alpha-2} = O(\nu^{-1});$$

it remains to prove the convergence of $\sum_1^\infty \nu^{-1}\mid s_\nu{}^{-\alpha}\mid$. Now, employing (1.6)

$$\sum_1^n \nu^{-1}\mid s_\nu{}^{-\alpha}\mid = n^{-1}\sigma_n + \sum_1^{n-1}\sigma_\nu(1/\nu(\nu+1)) = O(n^{-\alpha} + \sum_1^n \nu^{-1-\alpha}) = O(1);$$

thus all series in (3.1) are absolutely convergent, and

$$t^{-1}F(t) = \sum_1^\infty s_\nu{}^{-\alpha}T_\nu{}^\alpha(t).$$

We now choose a positive number μ, and put $n = [\mu t^{-1}]$,

$$\sum_1^\infty s_\nu{}^{-\alpha}T_\nu{}^\alpha = (\sum_1^n + \sum_{n+1}^\infty)s_\nu{}^{-\alpha}T_\nu{}^\alpha = U_n(t) + V_n(t), \quad \text{say.}$$

From (2.6)

$$\mid V_n(t)\mid < 2t^{-\alpha}\sum_{n+1}^\infty \nu^{-1}\mid s_\nu{}^{-\alpha}\mid,$$

and from (1.6)

$$\sum_{n+1}^\infty \nu^{-1}\mid s_\nu{}^{-\alpha}\mid = -(\sigma_n/(n+1)) + \sum_{n+1}^\infty \sigma_\nu(1/\nu(\nu+1)) < \sum_{n+1}^\infty \sigma_\nu(1/\nu(\nu+1))$$

$$= O(\sum_n^\infty \nu^{-1-\alpha}) = O(n^{-\alpha}) = \mu^{-\alpha}t^\alpha O(1);$$

hence

$$V_n(t) = \mu^{-\alpha}O(1).$$

Furthermore, in view of (2.5),

$$U_n = s_n{}^{1-\alpha}T_n{}^\alpha + \sum_1^{n-1} s_\nu{}^{1-\alpha}(T_\nu{}^\alpha - T_{\nu+1}{}^\alpha) = s_n{}^{1-\alpha}T_n{}^\alpha + \sum_1^{n-1} s_\nu{}^{1-\alpha}T_\nu{}^\alpha.$$

We may assume without loss of generality, that

$$s_n{}^{1-\alpha} = o(n^{1-\alpha});$$

then, employing (2.6)

$$U_n(t) = o(n^{-\alpha}t^{-\alpha}) + o(\sum_1^n \nu^{-\alpha}t^{1-\alpha}) = \mu^{-\alpha}o(1) + \mu^{1-\alpha}o(1).$$

Summarizing, we get

$$t^{-1}F(t) = \mu^{-\alpha}O(1) + \mu^{1-\alpha}o(1) \text{ as } t \to 0;$$

hence

$$\limsup_{t\to 0} t^{-1} \mid F(t) \mid \leq \mu^{-\alpha}O(1);$$

α being positive, μ arbitrarily large, and $O(1)$ independent of μ, we get finally

$$t^{-1}F(t) \to 0 \text{ as } t \to 0,$$

which proves our theorem. Note that the theorem becomes false for $\alpha = 0$. To illustrate the scope of the theorem, we construct a series summable $(C, 1-\alpha)$ and satisfying (1.6) but not summable (C, β) for any $\beta < 1-\alpha$. Put $s_n^{-\alpha} = k^{-\frac{1}{2}}2^{k(1-\alpha)}$ for $n = 2^k$, $k \geq 1$, and $s_n^{-\alpha} = 0$ otherwise. Then for $2^k \leq n < 2^{k+1}$

$$s_n^{1-\alpha} = \sum_1^n s_\nu^{-\alpha} = \sum_1^k \nu^{-\frac{1}{2}}2^{\nu(1-\alpha)} = o(2^{k(1-\alpha)}) = o(n^{1-\alpha});$$

hence $(C, 1-\alpha)\Sigma a_n = 0$. Moreover condition (1.6) is evidently satisfied, as $s_n^{-\alpha} \geq 0$ for all n; in fact $\sigma_n = o(n^{1-\alpha})$, which means strong summability $(C, 1-\alpha)$.

On the other hand, from (2.3) and (2.1) for $0 < \beta < \alpha < 1$

$$s_n^{\beta} = \sum_1^n \gamma_{n-\nu}\nu^{\beta+\alpha-1}s_\nu^{-\alpha} > s_n^{-\alpha};$$

hence, for $n = 2^k$,

$$n^{-\beta}s_n^{\beta} > k^{-\frac{1}{2}}2^{k(1-\alpha-\beta)} \to \infty,$$

so that our series is not summable (C, β).

4. Generalization of *L*-summability to integrals. Employing Stieltjes integrals the following generalization suggests itself:

Suppose that the integral

$$\phi(t) = \int_0^\infty u^{-1} \sin utdA(u) \text{ exists for } 0 < t < \tau, \tag{4.1}$$

and that

$$t^{-1}\phi(t) \to s \text{ as } t \to 0; \tag{4.2}$$

we then say that the integral $\int_0^\infty dA(u)$ is summable to s by the method ϕ.

If in particular $A(u)$ is a step function with the jump a_n at the point $u = n(1, 2, 3, \cdots)$, then $\phi(t) = \sum_1^\infty n^{-1}a_n \sin nt$, and our method reduces to the L-method.

If the jumps a_n occur at the points $u = \lambda_n$, where $\lambda_n \uparrow \infty$, then

$$\phi(t) = \sum_1^\infty \lambda_n^{-1} a_n \sin \lambda_n t. \tag{4.3}$$

We give now a generalization of L-summability to integrals without the use of Stieltjes integrals. By means of the identity

$$\cos(n - \tfrac{1}{2})t - \cos(n + \tfrac{1}{2})t = 2 \sin nt \sin(t/2)$$

(1.1) becomes

$$F(t) = \frac{1}{2 \sin t/2} \sum_1^\infty \frac{a_n}{n} \{\cos(n - \tfrac{1}{2})t - \cos(n + \tfrac{1}{2})t\};$$

but

$$\cos(n - \tfrac{1}{2})t - \cos(n + \tfrac{1}{2})t = t \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \sin utdu,$$

so that (1.2) is equivalent to

$$t^{-1} \sum_1^\infty n^{-1} a_n \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \sin utdu \to s. \tag{4.4}$$

We define

$$a(u) = 0 \text{ for } 0 \leq u < \tfrac{1}{2}, \quad a(u) = un^{-1}a_n \text{ for } n - \tfrac{1}{2} \leq u < n + \tfrac{1}{2}; \tag{4.5}$$

then (4.4) becomes

$$g(t) = t^{-1} \int_0^\infty u^{-1} a(u) \sin utdu \to s. \tag{4.6}$$

Thus the following definition also includes L-summability:

The integral $\int_0^\infty a(u)du$ is g-summable to s, if $\int_0^\infty u^{-1}a(u)\sin utdu$ exists in some interval $0 < t < \tau$, and if (4.6) holds.

In case of (4.5) we have the formal equation

$$\int_0^\infty a(u)du = \sum_1^\infty n^{-1}a_n \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} udu = \sum_1^\infty a_n.$$

5. Another generalization of L-summability to integrals. L-summability, as defined in **1**, is a series to function transform. We now show that it can be written as the sequence to function transform

$$\sum_1^\infty s_n \Delta \frac{\sin nt}{nt} \to s. \tag{5.1}$$

More precisely: if (1.2) holds, then the series (5.1) converges to the value $t^{-1}F(t)$, and conversely if (5.1) converges then $t^{-1}\Sigma n^{-1}a_n \sin nt$ converges to the same value.

To prove the first part, we employ a result due to B. Kuttner [3, §4], stating that (1.2) implies $s_n = o(n)$. Now

$$\sum_1^n \nu^{-1} a_\nu \sin \nu t = n^{-1} s_n \sin nt + \sum_1^{n-1} s_\nu \Delta \frac{\sin \nu t}{\nu} , \tag{5.2}$$

which yields the first part of our statement.

For the converse we shall use the lemma:

LEMMA 2. *If*

$$B_n(t) = a_n \sin nt + b_n \sin (n+1)t \to 0 \tag{5.3}$$

for every t belonging to a set $\mathcal{E}$ of positive measure, then $a_n \to 0$ and $b_n \to 0$.

This lemma granted, the convergence of the series (5.1) implies $s_n \Delta \frac{\sin nt}{nt} \to 0$; hence by the lemma $n^{-1} s_n \to 0$, and our statement follows from (5.2).

The lemma is an analogue to a well known theorem of Cantor and Lebesgue. For its proof put

$$\rho_n^2 = a_n^2 + b_n^2$$

where we may assume that $\rho_n > 0$. Clearly

$$B_n^2(t) < 2\rho_n^2;$$

hence, in view of (5.3)

$$\rho_n^{-1} B_n^2(t) < 2 \mid B_n(t) \mid \to 0 \text{ for } t \in \mathcal{E}.$$

Using a theorem of Egoroff, it follows that on a point set $\mathcal{E}' \subset \mathcal{E}$ of positive measure $\mid \mathcal{E}' \mid > 0$

$$\rho_n^{-1} B_n^2(t) \to 0 \text{ uniformly in } \mathcal{E}';$$

a fortiori

$$\rho_n^{-1} \int_{\mathcal{E}'} B_n^2(t)\, dt \to 0. \tag{5.4}$$

Now

$$2B_n^2 = a_n^2(1 - \cos 2nt) + b_n^2[1 - \cos 2(n+1)t] + 2a_n b_n[\cos t - \cos(2n+1)t];$$

hence

$$2 \int_{\mathcal{E}'} B_n^2(t)\, dt = (a_n^2 + b_n^2) \mid \mathcal{E}' \mid + 2a_n b_n \int_{\mathcal{E}'} \cos t\, dt + \mathcal{R}_n, \text{ say,}$$

and

$$2\rho_n^{-2} \int_{\mathcal{E}'} B_n^2(t)\, dt > \int_{\mathcal{E}'} (1 - (\cos t))\, dt - \rho_n^{-2} \mid \mathcal{R}_n \mid.$$

But $\rho_n^{-2}\mathcal{R}_n$ consists of three terms, each term being a Fourier coefficient of the characteristic function corresponding to $\mathcal{E}'$, multiplied by factors $<\pi$; hence $\rho_n^{-2}\mathcal{R}_n \to 0$ so that the sequence $\rho_n^{-2}\int_{\mathcal{E}'} B_n^2(t)\,dt$ is bounded away from zero. Combined with (5.4) this yields $\rho_n \to 0$, and our lemma is proved.

We may now write

$$t^{-1}F(t) = \sum_1^\infty s_n \Delta \frac{\sin nt}{nt} = t^{-1}\sum s_n \int_n^{n+1} u^{-2}(\sin ut - ut\cos ut)\,du;$$

let

$$A(u) = s_n \text{ for } n \leq u < n+1, \qquad (n = 1, 2, 3, \cdots),$$

then

$$F(t) = \int_1^\infty A(u)u^{-2}(\sin ut - ut\cos ut)\,du. \tag{5.5}$$

We are thus led to the following generalization of L-summability:

If the integral (5.5) exists in some interval $-\tau < t < \tau$, and if $t^{-1}F(t) \to s$ as $t \to 0$, then s is the generalized limit of $A(u)$.

The comparison of our integral transforms with Cesàro integral means will be discussed in another paper.

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ON A CONGRUENCE PROPERTY OF THE DIVISOR FUNCTION.*

By L. G. SATHE.

1. Introduction and notations. Let k be a given integer, r a fixed residue class (mod k) and $d(n)$ the number of divisors of n. Let $N(k, r, x)$ denote the number of $n \leq x$ for which $d(n) \equiv r \pmod{k}$. We are concerned in this paper with the asymptotic behavior of $N(k, r, x)$ as $x \to \infty$. S. S. Pillai [1] has proved that $N(k, 0, x) \sim Bx$ when k is a prime.[1] We here treat the problem for all values of k and r. The main result is:

THEOREM. *If 2^l is the highest power of 2 that divides k, then there is a constant $B = B(k, r) > 0$ such that*

$$N(k, r, x) \sim Bx \text{ if } 2^l | r \quad \text{and} \quad N(k, r, x) = o(x) \text{ if } 2^l \nmid r$$

The lemmas are given in **2** and the main theorem is proved in **3**. In **4** we investigate under what conditions $B(k, r)$ is equal to $B(k', r')$ and evaluate the constant B in special cases.

p is a prime, $f(n)$ denotes the number of different prime factors of n, $h(n) = \prod_{p|n} \{p/(p+1)\}$, $h(1) = 1$; u denotes a number such that if $p^\theta \| u$ then either (i) $\theta = 0$ or (ii) $2 \leq \theta \leq k-1$.[2] Thus every prime divisor of u occurs at least in the second power and at most in the $(k-1)$-th power. q always denotes a square free integer.

2. Lemmas.

LEMMA 1. *Let d be a given square free number and let $M(x, d, r) = M(x, d, r, e)$ be the number of square free integers $q \leq x$ for which $f(q) \equiv r \pmod{e}$ and $(q, d) = 1$. Then to every given positive number δ there corresponds a $\Delta = \Delta(\delta, e)$ independent of d and r such that*

$$|M(x, d, r) - Ah(d)x| < \delta 2^{f(d)} x \text{ if } x \geq \Delta \tag{1}$$

where $A = 1/e\zeta(2)$.

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[1] *Journal of the Indian Mathematical Society* vol. 6, N. S. (1942). The proof contains a mistake which has been pointed out and rectified by the present author, *Journal of the Indian Mathematical Society* vol. 7, N. S. (1943).

[2] $p^\theta \| n$ denotes that p^θ is the highest power of the prime p that divides n.

Proof. Let p be a given prime and let $M_1(x, r) = M_1(x, r, e)$ denote the number of $q \leq x$ for which $f(q) \equiv r \pmod{e}$. Then, obviously,

$$M(x, p, r) + M(x/p, p, r-1) = M_1(x, r). \tag{2}$$

In [2] and [3] it is proved that

$$M_1(x, r) \sim Ax \text{ for every fixed } r. \tag{3}$$

From (2) and (3) it follows that to every fixed r and arbitrarily small positive number δ there corresponds a number $\Delta' = \Delta'(\delta, e)$ independent of p and r such that

$$|M(x, p, r) - Ax + M(x/p, p, r-1)| < \tfrac{1}{2}\delta x \text{ if } x \geq \Delta'.$$

Hence,

$$\begin{aligned} &|M(x, p, r) - Ax + Ax/p - M(x/p^2, p, r-2)| \\ &\quad \leq |M(x, p, r) - Ax + M(x/p, p, r-1)| \\ &\qquad + |M(x/p, p, r-1) - Ax/p + M(x/p^2, p, r-2)| \\ &\quad < \tfrac{1}{2}\delta x + \tfrac{1}{2}\delta x/p \text{ if } x/p \geq \Delta'. \end{aligned}$$

Repeating this process we get finally, if $\nu = [\log x/2 \log p]$

$$\begin{aligned} &|M(x, p, r) - Ax + Ax/p + \cdots + (-1)^{\nu-1}Ax/p^{\nu} \\ &\quad + (-1)^{\nu}M(x/p^{\nu+1}, p, r-\nu-1)| < \tfrac{1}{2}\delta x(1 + 1/p + \cdots + 1/p^{\nu}) \\ &\quad \leq \tfrac{1}{2}\delta x \sum_{s=0}^{\infty} 1/2^s = \delta x \text{ if } x/p^{\nu} \geq \Delta', \text{ that is if } x \geq \Delta'^2. \end{aligned} \tag{4}$$

From (4) we get

$$\begin{aligned} |M(x, p, r) - Ax \sum_{s=0}^{\infty}(-1)^s/p^s| &< \delta x + Ax\{1/p^{\nu+1} + 1/p^{\nu+2} + \cdots\} + \sqrt{x} \\ &\leqq \delta x + Ax/p^{\nu+1}(1 + \tfrac{1}{2} + \cdots) + \sqrt{x} \\ &\leqq \delta x + 2A\sqrt{x} + \sqrt{x} \text{ if } x \geq \Delta'^2. \end{aligned}$$

Hence,

$$|M(x, p, r) - Ah(p)x| < \delta 2x \text{ if } x \geq \Delta = \Delta(\delta, e). \tag{5}$$

Suppose, now, that (1) is true for $d = d_j = \prod_{i=1}^{j} p_i$, $p_1, p_2, \cdots, p_j$ being different primes. We shall show that (1) is true for $d_{j+1} = \prod_{i=1}^{j+1} p_i$ also. We have evidently,

$$\begin{aligned} M(x, d_{j+1}, r) &= M(x, d_j, r) \\ &\quad - M(x/p_{j+1}, d_j, r-1) + M(x/p_{j+1}^2, d_j, r-2) - \cdots. \end{aligned}$$

Hence if $\mu = [\log x/2 \log p_{j+1}]$,

$$|M(x, d_{j+1}, r) - M(x, d_j, r) + \cdots + (-1)^{\mu+1} M(x/p_{j+1}^{\mu}, d_j, r-\mu)| \leq (x/p_{j+1}^{\mu+1})(1 + 1/p_{j+1} + \cdots) < 2\sqrt{x}.$$

By our suposition (1) is true for $d = d_j$. Therefore, to $\delta/2$ there exists a $\Delta'' = \Delta''(\delta, e)$ such that

$$|M(x, d_{j+1}, r) - Ah(d_j)x\{1 - 1/p_{j+1} + \cdots + (-1)^{\mu}(1/p_{j+1}^{\mu})\}| < \tfrac{1}{2}\delta x\, 2^{f(d_j)}\{1 + 1/p_{j+1} + \cdots + 1/p_{j+1}^{\mu}\} + 2\sqrt{x} \text{ if } x/p_{j+1}^{\mu} > \Delta'',$$

Δ'' being independent of $p_1, p_2, \cdots, p_{j+1}$. As before, since $h(n) \leq 1$,

$$|M(x, d_{j+1}, r) - Ah(d_j)x \sum_{s=0}^{\infty} (-1)^s/p_{j+1}^{s} < \delta 2^{f(d_j)}x + 2\sqrt{x} + 2A\sqrt{x} \text{ if } x \geq \Delta'^2.$$

Hence,

$$(6) \quad |M(x, d_{j+1}, r) - Ah(d_{j+1})x| < 2\delta_2 f^{(d_j)}x = \delta_2 f^{(d_{j+1})}x \text{ if } x \geq \Delta = \Delta(\delta, e).$$

From (5) and (6) the result follows by induction.

LEMMA 2. *The number of integers*

$$(7) \qquad m^k u \leq x$$

is $O(x^{1/2})$.

Proof. Let $q_2, q_3, \cdots$ denote square free integers. The number of integers q_2^2 is obviously $O(x^2/2)$. Hence the number of integers $q_2^2 q_3^3 \leq x$ where $(q_2, q_3) = 1$ is

$$O(\sum_{q_3 \leq x} \sqrt{x/q_3^3}) = O(x^{1/2}).$$

Proceeding in this way we see that the number of integers

$$(8) \qquad q_2^2 q_3^3 \cdots q_{k-1}^{k-1} \leq x, \qquad (q_i, q_j) = 1$$

is $O(x^{1/2})$ for any fixed $k \geq 3$.

Now u is either unity or an integer of the form (8), this representation for a given u being unique. Hence the number of $u \leq x$ is $O(x^{1/2})$. It follows that the number of integers of the form (7) is

$$O(x^{1/2} \sum_{m=1}^{\infty} 1/m^{k/2}) = O(x^{1/2}) \text{ if } k \geq 3.$$

If $k = 2$, the result is evident, since the set of u consists of one integer only, namely unity.

LEMMA 3. *If l is a fixed integer and q runs over those square free numbers for which $f(q) \leq l$ then the number of integers*

$$m^k qu \leq x \text{ where } (q,u)=1 \tag{9}$$

is $o(x)$ if $k \geq 2$.

Remark. We note that every integer n can be represented uniquely in the form

$$m^k qu \tag{10}$$

where $(q,u)=1$; m being the largest k-th power that divides n and q being the square free number formed by the product of different primes dividing n/m^k and occurring in the first power only. u is an integer of the type already defined. The lemma states that the number of integers $n \leq x$ for which q consists of not more than l primes is $o(x)$.

Proof. By virtue of Lemma (2) the contribution by a fixed q is $O(\sqrt{x/q})$. Hence the number of integers (9) is

$$O(x^{1/2} \sum_{q \leq x} 1/q^{1/2}). \tag{11}$$

Let $Q(x)$ be the number of $q \leq x$ with $f(q) \leq l$; then

$$Q(x) = \Pi_1(x) + \cdots + \Pi_l(x) = O(x/\sqrt{\log x}).^3 \tag{12}$$

Hence,

$$\begin{aligned} \sum_{q \leq x} 1/q^{1/2} &= \sum_{n \leq x} \{\phi(n) - \phi(n-1)\}/\sqrt{n} \\ &= \sum_{n \leq x} \phi(n)/\{1/\sqrt{n} - 1/\sqrt{n+1}\} + O(\phi(x)/\sqrt{x}) \\ &= O \sum_{2 \leq n \leq x} 1/\sqrt{n \log n} + o(x^{1/2}) \text{ by virtue of (12)} \\ &= O(\int_1^{\sqrt{x}} (dv/\sqrt{v}) + (1/\sqrt{\log x}) \int_{\sqrt{x}}^{x} (dv/\sqrt{v})) + o(x^{1/2}) \\ &= o(x^{1/2}). \end{aligned} \tag{13}$$

The lemma follows from (11) and (13).

LEMMA 4. *Let k be odd, r fixed and $\not\equiv 0 \pmod k$, $(r,k)=t$. Let e be the exponent of 2 $\pmod k$ and e' the exponent of 2 $\pmod{k/t}$. Then the numbers $r2^a$ where $a \geq 0$ belong to e' residue classes $\pmod k$ which are given by*

$$r2^\alpha \text{ where } 1 \leq \alpha \leq e'. \tag{14}$$

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[3] Landau, ***Primzahlen*** (pp. 205-211).

Moreover

(15) $\qquad r2^a \equiv r2^b \pmod{k}$ *if and only if* $a \equiv b \pmod{e'}$.

Proof. Trivial.

LEMMA 5. *Let k be odd, r fixed and $\not\equiv 0$; $(r, k) = t$. Let $n = m^k qu \leq x$ in the sense of* (10). *Suppose $f(q) \equiv \lambda' \pmod{e'}$. Then $n \subset N(k, r, x)$ if and only if*

(16) $\qquad d(u) \equiv r2^{e'-\lambda'} \pmod{k}$.

Proof. It is easily seen that $d(n) \equiv d(qu) \pmod{k}$ whether m is prime to qu or not. Hence, since $(q, u) = 1$,

$$d(n) \equiv d(q)d(u) \pmod{k}.$$

It follows that $n \subset N(k, r, x)$ if and only if

(17) $\qquad d(q)d(u) \equiv r \pmod{k}$.

Let $f(q) \equiv \lambda \pmod{e}$; then

(18) $\qquad d(q) = 2^{f(q)} \equiv 2^\lambda \pmod{k}$.

If now $n \subset N(k, r, x)$, then it follows from (17) that $d(u) \equiv r2^{e-\lambda} \pmod{k}$. Hence from (15), $d(u) \equiv r2^{e'-\lambda'} \pmod{k}$. Conversely, if $d(u) \equiv r2^{e'-\lambda'} \pmod{k}$, then it follows from (18) and (15) that $d(q)d(u) \equiv r2^{e'-\lambda'+\lambda} \equiv r \pmod{k}$, since $e' - \lambda' + \lambda \equiv 0 \pmod{e'}$.

Remark. If $d(u) \equiv r2^\alpha \pmod{k}$ and u is fixed then integers $n = m^k qu$ belong to $N(k, r, x)$ (for different values of m and q) if and only if

(19) $\qquad f(q) \equiv e' - \alpha \pmod{e'}$ and $(q, u) = 1$.

LEMMA 6. *If k is odd and $n = m^k qu \leq x$ then $n \subset N(k, 0, x)$ if, and only if, $d(u) \equiv o \pmod{k}$.*

Proof. This follows immediately from (17) which is true even if $r = 0$.

3. Proof of the main theorem.

THEOREM 1. *If k is odd, $1 \leq r \leq k - 1$, then $N(k, r, x) \sim Bx$.*

Proof. If $n = m^k qu \leq x$ where $d(u) \not\equiv r2^\alpha \pmod{k}$ then by Lemma 5 n does not belong to $N(k, r, x)$. Denote by u' an integer of the set u such that

(20) $\qquad d(u') \equiv r2^\alpha \pmod{k}$,

where we may regard $1 \leq \alpha \leq e'$ by virtue of Lemma 4. From (19) we

see that the contribution to $N(k, r, x)$ *by a fixed* u' is equal to the number of integers

(21) $\quad m^k q u' \leq x$ where $f(q) \equiv e' - \alpha \pmod{e'}$ and $(q, u') = 1$.

This, in the notation of Lemma 1, is equal to

$$(22) \qquad \sum_{m^k \leq x/u'} M(x/m^k u', \prod_{p|u'} p, e' - \alpha, e') = \sum_{m^k \leq x^{7/8}/u'} + \sum_{x^{7/8}/u' < m^k \leq x/u'} = S_1 + S_2, \text{ say.}$$

$S_2 = O(x^{1/8+1/k}) = O(x^{11/24})$ since $k \geq 3$. Suppose that the fixed number u' is not greater than $x^{3/4}$. Then from Lemma 1, since $h(\prod_{p|u'} p) = h(u')$ and $x/m^k u' > x^{1/8}$, there exists, for every positive number δ, a $\Delta = \Delta(\delta, e')$ independent of u' such that

$$| S_1 - \sum_{m^k \leq x^{7/8}/u'} x h(u')/e'\zeta(2) m^k u' | < \delta 2^{f(u')} x/u' \sum_{m=1} 1/m^k \text{ if } x \geq \Delta^8.$$

Hence, since $h(u') \leq 1$,

$$| S_1 - x h(u')\zeta(k)/e'\zeta(2)u' | < \delta\zeta(k) 2^{f(u')} x/u' + x/e'\zeta(2)u' \sum_{m > x^{1/8k}}^{\infty} 1/m^k \text{ if } x \geq \Delta^8$$

$$< \delta(2^{f(u')}/u')x\zeta(k) + \delta x\zeta(k)/u' \text{ if } x \geq \Delta_0$$

Δ_0 being independent of u'. Hence,

$S_1 = x h(u')\zeta(k)/e'\zeta(2)u' + O(\delta(2^{f(u')}/u')x)$ for every arbitrarily small δ where the constant in O is independent of δ and u'. From this and from (22) the contribution to (21) by every fixed $u' \leq x^{3/4}$ is equal to

$$(23) \qquad x h(u')\zeta(k)/e'\zeta(2)u' + O(\delta x 2^{f(u')}/u') + O(x^{11/24}).$$

Now the total contribution to (21) by all $u' > x^{3/4}$ is $O(x^{3/4})$ since the contribution by every such u' is at most equal to $x^{1/4}$ and the number of $u' \leq x$ is $O(x^{1/2})$ by Lemma 2. From this and from (19) we get,

$$N(k, r, x) = x\zeta(k)/e'\zeta(2) \sum_{u' \leq x^{3/4}} h(u')/u' + O(\delta x \sum_{u' \leq x^{3/4}} 2^{f'(u)}/u') + O(x^{11/24} \sum_{u' \leq x^{3/4}}) + O(x^{3/4}).$$

The series $\sum_{u'} h(u')/u'$ and $\sum_{u'} 2^{f(u')}/u'$ are both convergent; for if $R(x)$ be the number of $u \leq x$ then,

$$\sum_{u'} 1/u'^a \leq \sum_{u} 1/u^a = O \sum_{n=1}^{\infty} \{R(n) - R(n-1)\}/n^a = O \sum_{n=1}^{\infty} R(n)/n^{1+a}$$

$$= o \sum_{n=1}^{\infty} 1/n^{a+1/2} \text{ by Lemma 2.}$$

Hence $\Sigma 1/u'^a$ converges if $a > 1/2$. From this the convergence of the series in question follows since $h(u') \leq 1$ and $2^{f(u')} \leq d(u') = O(u'^\epsilon)$ for every $\epsilon > 0$. Finally,

$$N(k, r, x) = x\zeta(k)/e'\zeta(2) \sum_{u'} h(u')/u' + O(\delta x) + o(x)$$
$$\sim Bx \text{ for every positive } \delta$$

(since the constant in $O(\delta x)$ is independent of δ) where,

$$B = C\zeta(k)/e'\zeta(2) \quad \text{and} \quad C = \sum_{u'} h(u')/u', \tag{24}$$

u' running over all the integers u satisfying (20). To show that $B > 0$ we have only to prove that $C > 0$; that is that the set u' is not empty. Evidently if p is any prime, then p^{r-1} is a u' if $k - 1 \geq r \geq 3$. If $r = 1$ or 2 we see that 1 is a u'. For $d(1) \equiv 1 \pmod{k}$ and $d(1) \equiv 2\ 2^{e'-1} \equiv 1 \pmod{k}$. Hence (20) is satisfied for $u' = 1$.

THEOREM 2. *If* $k = 2^l k'$; $(2, k') = 1$

$$r = 2^t m'; \ (2, m') = 1; \ 0 \leq t \leq l - 1$$

then $N(k, r, x) = o(x)$.

Proof. If $n = m^k qu \leq x$ then, as in (17), n belongs to $N(k, r, x)$ if and only if $d(q)d(u) \equiv 2^t m' \pmod{2^l k'}$. Dividing this congruence by 2^t we see that it has no solution with $f(q) \geq l$ if $t \leq l - 1$. The result, therefore, follows from Lemma 3.

THEOREM 3. $N(2^l k', 2^l m', x) \sim Bx$, $B > 0$, *where* k' *is odd and* $1 \leq m' \leq k' - 1$.

Proof. Let

$$2^l m' \equiv m'' \pmod{k'}. \tag{25}$$

We note in the first instance that if

$$n \subset N(2^l k', 2^l m', x) \quad \text{then} \quad n \subset N(k', m'', x). \tag{26}$$

Also if $n \subset N(k', m'', x)$ then $d(n) \equiv m'' \pmod{k'}$. Hence from (25), $d(n) \equiv zk' + 2^l m' \pmod{2^l k'}$ where $z = 0, 1, \cdots$ or $2^l - 1$. Hence either

(27) $d(n) \equiv 2^l m' \pmod{2^l k'}$ or $d(n) \equiv s \pmod{2^l k'}$ where $2^l \nmid s$.

From (26) and (27) it follows that

$$N(k', m'', x) = N(2^l k', 2^l m', x) + X$$

where $X = O \sum_s N(2^l k', s, x)$, s running through all the residue classes $(\text{mod } 2^l k')$ for which $2^l \nmid s$. Hence $X = o(x)$ by Theorem 2. Therefore,

(28) $$N(2^l k', 2^l m', x) \sim N(k', m'', x) \sim Bx$$

by Theorem 1, where $B > 0$.

THEOREM 4. $N(k, 0, x) \sim Bx$.

Proof. It is sufficient to consider the case when k is odd, since it *then* follows as in (28) that

(28′) $$N(2^l k', 0, x) \sim N(k', 0, x) \text{ if } k' \text{ is odd.}$$

It follows from Lemma 6 that integers n belonging to $N(k, 0, x)$ are of the form

(29) $$m^k q u'' \leq x \text{ where } (q, u'') = 1,$$

q being any square free integer, u'' being a u for which $d(u) \equiv 0 \pmod{k}$. If now $Q(x, d)$ denotes the number of square free integers $q \leq x$ for which $(q, d) = 1$ then, from Lemma 1

(30) $$Q(x, d) = M(x, d, 1) + \cdots + M(x, d, e) = h(d)x/\zeta(2) + O(\delta 2^{f(d)}x)$$

for every $\delta > 0$, the constant in $O(\delta 2^{f(d)}x)$ being independent of δ and d. Also

$$N(k, 0, x) = \sum_{m^k u'' \leq x} Q(x/m^k u'', \prod_{p|u''} p)$$

the summation being with respect to both m and u''. Using (30) we get as in Theorem 1

$$N(k, r, x) \sim Bx$$

where

(31) $$B = \zeta(k)/\zeta(2) \sum_{u''} h(u'')/u'',$$

where u'' runs over all the integers of the set u for which $d(u) \equiv 0 \pmod{k}$. To prove that $B > 0$ we have to show that the set u'' is not empty. Evidently p^{k-1} is a u'' if p is any prime.

The proof of the main theorem now follows from Theorems 1 to 4.

4. Miscellaneoous results.

THEOREM 5. *Let k be odd; $(r, k) = t$ and, as in Lemma 4, let e' be the exponent of 2 (mod k/t). Then,*

(32) $$N(2^l k, 2^l r, x) \sim N(k, r2^\beta, x); \qquad 1 \leq \beta \leq e'.$$

In particular when $l = 0$ and 2 is the primitive root of k, we get $N(k, r, x) \sim N(k, r', x)$ if neither r nor r' is zero.

Proof. Let k be odd; $1 \leq r \leq k-1$. We note in the first instance that only those u' which satisfy (20) contribute to $N(k, r2^\beta, x)$. Also (23) still remains true since the dominant term of (22) is independent of α. Hence we see that (24) is true for $r2^\beta$ also. Hence (32) is true for odd values of k and all values of r, since when $r=0$ it reduces to an identity.

If now $k=2^l$, $(k', 2)=1$ then we get from (28) that

$$N(2^l k', 2^l m', x) \sim N(k', m'', x) \tag{33}$$

where $m'' \equiv 2^l m' \pmod{k'}$. But by the case just considered,

$$N(k', m'', x) \sim N(k', m''2^\gamma, x) \text{ where } 1 \leq \gamma \leq e'.$$

Also $m'2^\beta$ and $m''2^\gamma$, where $1 \leq \beta \leq e'$; $1 \leq \gamma \leq e'$, run over the same classes of residues (mod k') by virtue of (25). Hence,

$$N(2^l k', 2^l m', x) \sim N(k', m'2^\beta, x) \text{ where } 1 \leq \beta \leq e'.$$

Writing k for k' and r for m' we get (32) for $1 \leq r \leq k-1$. If $r=0$ then (32) reduces to (28′). Hence the theorem is proved.

THEOREM 6.

$$N(2^l k, 0, x) \sim \{1 - \zeta(k)/\zeta(k-1)\}x, \text{ if } k \text{ is a prime.} \tag{34}$$

Proof. By virtue of (28′) we may suppose $l=0$ without loss of generality. To prove the result we must evaluate the sum of the series $\sum_{u''} h(u'')/u''$ where u'' runs over those u for which $d(u) \equiv 0 \pmod{k}$. Since k is a prime, every u'' is of the form

$$q_2^2 \cdots k_{k-2}^{k-2} q'^{k-1}_{k-1}$$

where $q_2, \cdots, q_{k-2}$ run over all square free numbers and q'_{k-1} runs over all square free numbers except unity. $q_2, \cdots, q'_{k-1}$ in every such number are relatively prime. Hence,

$$\sum_{u''} h(u'')/u'' = \sum_{q_2, \cdots, q'_{k-1}} (1/q'^{k-2}_{k-2})\Big(\prod_{p|q'_{k-1}} 1/(p+1) \cdots (1/q_2)\Big(\prod_{p|q_2} 1/(p+1)\Big).$$

In every term of this multiple series $q_2, \cdots, q'_{k-1}$ are mutually prime. Hence fixing $q_2, \cdots, q'_{k-1}$ and summing first with respect to q_2 we get

$$\begin{aligned}\sum (1/q_2)\Big(\prod_{p|q_2} 1/(p+1)\Big) &= \prod_p \{1 + 1/(p+1)\} \Big/ \prod_{p|q_2 \cdots q'_{k-1}} \{1 + 1/p(p+1)\} \\ &= \{\prod_p (1 - 1/p^3)\} \prod_{p|q_2 \cdots q'_{k-1}} \{p(p+1)/(p^2+p+1)\} \Big/ \prod_p (1 - 1/p^2) \\ &= \zeta(2)/\zeta(3) \prod_{p|q_2 \cdots q'_{k-1}} \{p(p+1)/(p^2+p+1)\}.\end{aligned}$$

Hence the series summation with respect to q_2 reduces to

$$\zeta(2)/\zeta(3)\sum(1/q'_{k-2}{}^{k-3})\Big(\prod_{p|q'_{k-1}} 1/(p^2+p+1)\Big)\cdots(1/q_3)\Big(\prod_{p|q_3} 1/(p^2+p+1)\Big).$$

We now fix $q_4, \cdots, q'_{k-1}$ and sum with respect to q_3 and so on. Finally, noting that q'_{k-1} does not take the value 1 we get

$$(35)\quad \sum h(u'')/u'' = \frac{\zeta(2)}{\zeta(3)}\cdot\frac{\zeta(3)}{\zeta(4)}\cdots\frac{\zeta(k-2)}{\zeta(k-1)}\left\{\frac{\zeta(k-1)}{\zeta(k)}-1\right\}$$

$$= \frac{\zeta(2)}{\zeta(k-1)}\left\{\frac{\zeta(k-1)}{\zeta(k)}-1\right\}.$$

From (35) and (31) the theorem follows.

Note. Let 2^t be the highest power of 2 that divides k and $1 \leq t \leq l-1$. Then, by sharpening Lemma 4 we can prove that

$$N(k, r, x) \sim D(x/\log x)(\log\log x)^{t-1} \text{ if } 2^t \| r$$

where $D = D(k, r)$ is a constant > 0. Further if r is odd then it is easy to see that $N(k, r, x) = O(x^{1/2})$, *if k is even.*

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[2] S. S. Pillai, *Proceedings of the Indian Academy of Sciences*, vol. 11 (1940).
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THE RECTILINEAR MOTION OF A GAS II.*

By MONROE H. MARTIN.

In the rectilinear motion of a gas we face the problem of determining the velocity u, the density ρ and the pressure p as a function of the position x and time t. In addition to meeting any initial conditions, which may be imposed on the motion of the gas, these three functions must satisfy the underdetermined system of partial differential equations

$$(1) \qquad (u_t + u_x u)\rho + p_x = 0, \qquad \rho_t + (\rho u)_x = 0.$$

If the second equation in (1) is multiplied by u and added to the first, the system (1) is seen to be equivalent to the system

$$(2) \qquad (\rho u)_t + (p + u^2\rho)_x = 0, \qquad \rho_t + (u\rho)_x = 0.$$

Assuming that a solution u, ρ, p of this system is at hand, there exist functions v, z of x, t such that the system of Pfaff

$$dv = -u\rho dx + (p + u^2\rho)dt, \qquad dz = \rho dx - u\rho dt,$$

possesses a solution in which the variables u, ρ, p, z, v are all functions of x, t. Conversely, given a solution of the system of Pfaff in which u, ρ, p, z, v are functions of x, t, one has in u, ρ, p a solution of (2), and hence a solution of (1).

The usual procedure is to make (1) a determined system by adding the characteristic equation $p = p(\rho)$ of the gas. The question has been raised,[1] however, whether the characteristic equation, known to hold for a perfect gas at rest, will continue to hold for a gas in motion. In this note, instead of assuming a characteristic equation for the gas, we supplement (1) by a third partial differential equation for u, p. This partial differential equation represents the necessary and sufficient condition that the Principle of Conservation of Energy hold for a gas in motion.

According to thermodynamical principles[2] the internal energy e per unit mass of a perfect gas is given by $e = p/(\gamma - 1)\rho$, where γ denotes the ratio of the two specific heats of the gas. The total energy E of the gas in a right

* Received September 9, 1944.

[1] H. Weber, "*Die partiellen Differentialgleichungen der Mathematischen Physik*," Braunschweig, vol. 2 (1919), p. 553.

[2] H. Weber, *op. cit.*, p. 507.

circular cylindrical tube of unit cross-sectional area, with bases on the planes $x = x_1$, $x = x_2$, is accordingly

$$E = \int_{x_1}^{x_2} (\tfrac{1}{2}u^2 + e)\rho dx, \qquad x_1 < x_2.$$

The Principle of Conservation of Energy holds if, and only if, the rate at which energy is being lost from the tube equals the rate at which the two bases of the tube do work on the surrounding gas,[3] that is

$$\dot{E} = p_1u_1 - p_2u_2 = -\int_{x_1}^{x_2} (pu)_x dx,$$

where here and elsewhere the subscripts 1, 2 on a variable indicate the values of this variable on the planes $x = x_1$, $x = x_2$. Using the above expression for E, it may be shown that this equation holds in conjunction with (1) if, and only if, u, p satisfy the partial differential equation

$$p_t + up_x + \gamma pu_x = 0.$$

If we set $p = \sigma^\gamma$, this equation becomes $\sigma_t + (\sigma u)_x = 0$, analogous to the equation of continuity in (1), and we have a determined system of partial differential equations

$$(u_t + u_x u)\rho + p_x = 0, \qquad \rho_t + (u\rho)_x = 0, \qquad \sigma_t + (u\sigma)_x = 0,$$

for the unknown functions u, ρ, σ. As above, we see that the problem of finding a solution $u, \rho, p = \sigma^\gamma$ of this system is equivalent to the problem of finding functions u, ρ, σ, v, z, w of x, t satisfying the system of Pfaff

$$dv = -u\rho dx + (\sigma^\gamma + u^2\rho)dt, \quad dz = \rho dx - u\rho dt, \quad dw = \sigma dx - u\sigma dt.$$

It will be observed that $\sigma dz - \rho dw = 0$, so that the functions z, w of x, t are always functionally dependent, $w = w(z)$. Now $\sigma = w'\rho$, where w' denotes the derivative of w with respect to z. Thus we have $p = (w'\rho)^\gamma$ and the above system of Pfaff may be replaced by a system of two equations of Pfaff in six variables x, t, u, ρ, z, v,

$$(3) \qquad dv = -udz + (w'\rho)^\gamma dt, \qquad dz = \rho dx - u\rho dt,$$

containing an arbitrary function $w' = w'(z)$.

The difference $z_2 - z_1$ of z for fixed t yields the mass of the gas in the cylindrical tube at the instant t while $v_1 - v_2$ yields its momentum. Fixing our attention on a particular gas particle, the value of z assigned to this gas

[3] Naturally this involves the hypothesis that there is no loss of energy by heat conduction or radiation.

particle remains constant throughout the motion of the gas. *The trajectories of gas particles in the (x, t) plane are accordingly represented by the one parameter family of curves $z = const.$*

Since z is constant along a trajectory, the relation between p and ρ along a trajectory follows the adiabatic law $p = k^2\rho^\gamma$, with $k^2 = w'^\gamma$ varying from trajectory to trajectory unless $w' = \text{const}$. From thermodynamical considerations it is known [2] that $S = c_v \log k^2$, where S denotes the entropy and c_v the specific heat of the gas at constant volume. Hence $S = \gamma c_v \log w'$ and if we assume $c_v = \text{constant}$ throughout the gas, we have the following result. *The entropy remains constant along a trajectory and varies from trajectory to trajectory with $\log w'$.*

We observe that the first equation in (3) is of class 5 and assumes normal form if we set $p = (w'\rho)^\gamma$, $q = -u$. Accordingly (3) is equivalent to the system of Pfaff

$$(4) \qquad dv = pdt + qdz, \qquad dx = -qdt + w'p^n dz, \qquad n = -1/\gamma,$$

in the six variables t, z, v, p, q, x.

Given functions p, q, v, x of z, t satisfying (4), we have $p = v_t$, $q = v_z$ and therefore, from the second equation of (4), v is a solution of the partial differential equation

$$(5) \qquad v_{zz} + nw'v_t^{n-1}v_{tt} = 0.$$

Conversely starting with a solution $v = v(z, t)$ of (5), we find u, ρ, p expressed in terms of z, t by

$$(6) \qquad u = -v_z, \qquad \rho = 1/w'v_t^n, \qquad p = v_t.$$

To express u, ρ, p in terms of x, t, one solves

$$x = x(z, t) = \int \{-v_z dt + w'v_t^n dz\},$$

for z as a function of x, t.

If we take u, ρ, p as known functions of x for $t = 0$ and assume $\rho > 0$ everywhere, $z = \int \rho dx$ is a monotonic increasing function of x. Accordingly we may regard u, ρ, p as known functions of z for $t = 0$. From (6) it appears that not only are the values of v_t, v_z prescribed for $t = 0$, but also the function $w'(z) > 0$ appearing in (5) is uniquely determined. *The problem of determining the rectilinear motion of a gas subsequent to prescribed initial conditions on its velocity, density, and pressure is accordingly reduced to the solution of the Problem of Cauchy for the partial differential equation* (5),

the function $w' = w'(z)$ having previously been determined from the initial conditions.

Once w' is determined, solutions of (5) may be obtained by the usual process of separation of variables. Thus $v = TZ$ is a solution, provided $T = T(t)$, $Z = Z(z)$ are solutions of

$$Z'' - knw'Z^n = 0, \qquad T'^{n-1}T'' + kT = 0, \qquad k = \text{const.}$$

In the special case where the entropy is constant throughout the gas ($w' = \text{const.}$), Legendre's transformation transforms (5) into a linear partial differential equation of second order and hyperbolic type. After reduction to normal form, Riemann's method may be applied to obtain a solution for the Problem of Cauchy set by the initial conditions.[4]

It may be of some interest to point out that in the general case Ampère's contact transformation

$$t = -P, \qquad z = Z, \qquad v = V - PT, \qquad p = T, \qquad q = Q,$$

carries (5) into a Monge-Ampère partial differential equation

$$V_{TT}V_{ZZ} - V^2_{ZT} + \lambda^2 = 0, \qquad \lambda^2 = -nT^{n-1}w'.$$

The characteristics of this partial differential equation comprise the asymptotic lines on the solution surfaces.[5]

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[4] See B. Riemann, "Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite," *Werke*, Leipzig (1892), 2nd Edition, pp. 156-175. Further references are given in the paper by the author, "The rectilinear motion of a gas," *American Journal of Mathematics*, vol. 65 (1943), pp. 391-407.

[5] E. Goursat, *Leçons sur l'integration des équations aux dérivées partielles du second ordre*, vol. 1, Paris (1896), pp. 147-148.

ON THE ASYMPTOTIC BEHAVIOR OF THE RIEMANN ξ-FUNCTION.*

By E. K. Haviland.

As is well known, the function, $\xi(s)$, defined by

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s),$$

possesses, as its only zeros, the non-trivial zeros of the Riemann ζ-function.[1] Furthermore, if $M(r)$ denotes the maximum of $|\xi(\frac{1}{2}+is)|$ on the circle $|s| = r$, then

$$\log M(r) \sim \tfrac{1}{2}r\log r. \tag{1}$$

This asymptotic relation is a well-known result of Hadamard and is more than sufficient for his purpose in applying his theory of entire functions to the Riemann ξ-function.[2] Actually, he defines $M(r)$ to be the maximum of $|\xi(s)|$ on the circle $|s| = r$, which does not affect the formula (1), but would affect the more delicate relations to be considered below.

There arises the desirability of an asymptotic formula for $M(r)$ itself, since the situation, as observed by Wintner, is as follows: Every *individual* zero of ξ has a number-theoretical significance (the explicit meaning of which is unknown). On the other hand, it is clear from the Weierstrass-Hadamard factorization

$$\xi(\tfrac{1}{2}+is) = \prod_{\rho}(1 - s^2/\rho^2) \tag{2}$$

that the omission or the "wrong" position (i. e., not in accordance with Riemann's hypothesis) of a finite number, or even of a sufficiently "thin" infinite sequence, of zeros has no influence whatsoever on (1), while each of these individual zeros will make a contribution which appears in the asymptotic behavior of $M(r)$, if this behavior is not weakened to standards of the logarithmic scale.

In fact, it turns out that the true law to which the classical result (1) is the logarithmic approximation is

* Received April 10, 1945.

[1] Cf., e. g., A. E. Ingham, *op. cit.*, p. 48.

[2] *Ibid.*, pp. 56-58.

$$M(r) \sim (\tfrac{1}{2}\pi)^{1/4}(2\pi e)^{-\frac{1}{2}r}r^{\frac{1}{2}r+7/4}, \quad \text{as} \quad r \to \infty. \tag{3}$$

But it is clear from (2) that the contribution of each of the factors to $M(r)$ is $r^2/|\rho|^2$, so that the contribution of each of the zeros to the right-hand side of (3) is explicit indeed. Incidentally, the 7/4 and the other elements occurring in (3) may be thought of as manifestations of a Fourier transformation of the corresponding elements in the Riemann–v. Mangoldt formula. In this connection, see the remarks of Wintner.[3]

Actually, (3) can be improved by showing the existence of an infinite asymptotic series

$$M(r) \sim (\tfrac{1}{2}\pi)^{1/4}(2\pi e)^{-\frac{1}{2}r}r^{\frac{1}{2}r+7/4}\sum_{n=0}^{\infty} C_n/r^n, \tag{4}$$

where the C's are numerical coefficients and (3) means merely that the numerical value of C_0 is 1.

For the purposes of carrying out the proof of these results, it is more convenient to deal, not with $\xi(s)$ as above defined, but with

$$\Xi(t) = \xi(\tfrac{1}{2} + it), \qquad t \text{ complex},$$

which is $\xi(t)$ in the notation of Riemann. For $\Xi(t)$, one has the formula:[4]

$$\Xi(t) = 2\int_0^{\infty} \Phi(u)\cos 2ut\, du,$$

where

$$\Phi(u) = 4\sum_{n=1}^{\infty}(2n^4\pi^2e^{9u} - 3n^2\pi e^{5u})\exp[-n^2\pi e^{4u}],$$

and it is known that $\Phi(u)$ is an even function of u, positive for all real values of u and such that $\Phi'(u) < 0$ for every $u > 0$.[5]

Corresponding to this integral representation of $\Xi(t)$, it is convenient to re-define $M(r)$ so as to denote

$$M(r) = \operatorname*{Max}_{|t|\leq r}|\Xi(t)| = \Xi(ir) = 2\int_0^{\infty}\Phi(u)\cosh 2ur\, du \tag{5}$$

$$= \int_0^{\infty}\Phi(u)e^{2ru}du + \int_0^{\infty}\Phi(u)e^{-2ru}du = I + II.$$

and our problem is to examine the behavior of $M(r)$, as $r \to \infty$.

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[3] A. Wintner, *loc. cit.* 5, p. 58.

[4] Cf., e. g., E. C. Titchmarsh, *op. cit.*, p. 45.

[5] Cf. A. Wintner, *loc. cit.* 4, p. 82, and further references there given.

We begin by writing

$$\Phi(u) = 4\sum_{n=1}^{k}(2n^4\pi^2e^{9u} - 3n^2\pi e^{5u})\exp[-n^2\pi e^{4u}] + 4\sum_{n=k+1}^{\infty}(2n^4\pi^2e^{9u} - 3n^2\pi e^{5u})\exp[-n^2\pi e^{4u}] = \Phi_k(u) + \Psi_k(u), \tag{6}$$

where $0 \leq u < +\infty$. In the foregoing infinite series, we replace e^u by x, so that it becomes

$$4\sum_{n=k+1}^{\infty}(2n^4\pi^2x^9 - 3n^2\pi x^5)\exp[-n^2\pi x^4], \qquad 1 \leq x < +\infty.$$

Since $2n^2\pi > 3$, $n \geq 1$, it follows that

(i) each term is positive and hence the series, if convergent, is positive for all x in the range indicated;

(ii) the series is term for term less than

$$8x\sum_{n=k+1}^{\infty}n^4(\pi x^4)^2\exp[-n^2\pi x^4].$$

Under the substitution $\pi x^4 = y$, this last series becomes

$$8(y/\pi)^{1/4}\sum_{n=k+1}^{\infty}n^4y^2\exp[-n^2y], \qquad \pi \leq y < +\infty. \tag{7}$$

Since $d(x^2e^{-x})/dx = xe^{-x}(2-x) < 0$ for $x > 2$, it follows that, for fixed $y \geq \pi$ and for $n \geq 2$, the general term of (7) is a positive monotone decreasing function of n, so that

$$\sum_{n=k+1}^{\infty}n^4y^2\exp[-n^2y] < (k+1)^4y^2\exp[-(k+1)^2y] + \int_{k+1}^{\infty}y^2s^4\exp[-ys^2]ds,$$

$k \geq 1$. This last integral is less than

$$\int_{k+1}^{\infty}y^2s^5\exp[-ys^2]ds = 1/2y\int_{y(k+1)^2}^{\infty}x^2e^{-x}dx, \quad \text{where} \quad x = ys^2.$$

Partial integration then shows that

$$\sum_{n=k+1}^{\infty}n^4y^2\exp[-n^2y] < (k+1)^4y^2\exp[-(k+1)^2y] + (2y)^{-1}\exp[-(k+1)^2y]\{(k+1)^4y^2 + 2(k+1)^2y + 2\} < 2(k+1)^4y^2\exp[-(k+1)^2y],$$

in view of $k \geq 1$, $y \geq \pi$. Consequently,

$$(8) \quad 0 < \Psi_k(u) \equiv \Psi_k(u(y)) < 16(y/\pi)^{1/4}(k+1)^4 y^2 \exp[-(k+1)^2 y],$$

and so

$$\begin{aligned}\int_0^\infty \Psi_k(u) e^{2ru} du &= \tfrac{1}{4}\pi^{-\frac{1}{2}r} \int_\pi^\infty \Psi_k(u(y)) y^{\frac{1}{2}r-1} dy \\ &< 4(k+1)^4 \pi^{-\frac{1}{2}r-1/4} \int_\pi^\infty y^{\frac{1}{2}r+5/4} \exp[-(k+1)^2 y] dy \\ &= 4[\pi(k+1)^2]^{-\frac{1}{2}r-1/4} \int_{(k+1)^2\pi}^\infty t^{\frac{1}{2}r+5/4} e^{-t} dt \\ &< 4[\pi(k+1)^2]^{-\frac{1}{2}r-1/4} \Gamma(\tfrac{1}{2}r + 9/4),\end{aligned}$$

where $(k+1)^2 y = t$.

With the help of the preceding inequalities, it may now readily be shown[6] that $\int_0^\infty \Phi(u) e^{2ru} du$ may, for fixed r, be integrated term by term. Moreover, as we have just seen,

$$(9) \quad \int_0^\infty \Psi_k(u) e^{2ru} du < [\pi(k+1)^2]^{-\frac{1}{2}r-1/4} (r + 5/2)(r + \tfrac{1}{2}) \Gamma(\tfrac{1}{2}r + 1/4).$$

In the finite set of terms obtained from $\int_0^\infty \Phi_k(u) e^{2ru} du$, that one will contribute least which corresponds to the term

$$-12k^2\pi e^{5u} \exp[-k^2\pi e^{4u}]$$

of $\Phi_k(u)$, for such a term is in absolute value less than the corresponding positive term in (6) and

$$d(x^2 \exp[-ax^2])/dx = 2x \exp[-ax^2](1 - ax^2) < 0,$$

if $a > 1$, $x > 1$, as is here the case, since $a = \pi e^{4u}$, $u \geq 0$ and $k \geq 1$. Then let

$$B_k(r) = -12k^2\pi \int_0^\infty e^{2ru+5u} \exp[-k^2\pi e^{4u}] du.$$

If we make the substitution $\pi e^{4u} = y$, we obtain

$$B_k(r) = -3k^2\pi^{-\frac{1}{2}r-1/4} \int_\pi^\infty y^{\frac{1}{2}r+1/4} \exp[-k^2 y] dy.$$

The further substitution $k^2 y = t$ leads to

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[6] Cf., e. g., W. F. Osgood, *op. cit.*, p. 287.

$$B_k(r) = -3(\pi k^2)^{-\frac{1}{2}r-1/4}\int_{\pi k^2}^{\infty} t^{\frac{1}{2}r+1/4}e^{-t}dt$$

$$= -3(\pi k^2)^{-\frac{1}{2}r-1/4}\int_0^{\infty} t^{\frac{1}{2}r+1/4}e^{-t}dt + o(1),$$

since

$$0 < (\pi k^2)^{-\frac{1}{2}r-1/4}\int_0^{\pi k^2} t^{\frac{1}{2}r+1/4}e^{-t}dt < (\pi k^2)^{-\frac{1}{2}r-1/4}\cdot(\pi k^2)^{\frac{1}{2}r+5/4}/(\tfrac{1}{2}r+5/4)$$

$$= \pi k^2/(\tfrac{1}{2}r+5/4) = o(1), \quad \text{as} \quad r\to\infty.$$

Hence

$$B_k(r) = -3/2(\pi k^2)^{-\frac{1}{2}r-1/4}(r+\tfrac{1}{2})\Gamma(\tfrac{1}{2}r+1/4) + o(1). \tag{10}$$

Comparison of (10) with (9) shows that

$$\int_0^{\infty}\Psi_k(u)e^{2ru}du = o(B_k(r)), \quad \text{as} \quad r\to\infty.$$

Furthermore, on setting $k=1$ in (8) and observing that $y=\pi e^{4u}$, we find that

$$0 < \Psi_1(u) < 256\pi^2 e^{9u}\exp[-4\pi e^{4u}].$$

This, together with (6) and (5), shows that II is $O(1)$, hence $o(B_k(r))$, for every fixed k, as $r\to\infty$.

Consequently, it is possible to obtain an expression for $M(r)$ in terms of an infinite asymptotic series. In carrying out this process, we first calculate

$$A_k(r) = 8k^4\pi^2\int_0^{\infty} e^{2ru+9u}\exp[-k^2\pi e^{4u}]du.$$

The calculation, which is exactly similar to that for $B_k(r)$ above, shows that

$$A_k(r) = \tfrac{1}{2}(\pi k^2)^{-\frac{1}{2}r-1/4}(r+5/2)(r+\tfrac{1}{2})\Gamma(\tfrac{1}{2}r+1/4) + o(1), \quad \text{as} \quad r\to\infty.$$

Hence, if $k=1$,

$$\int_0^{\infty}\Phi_1(u)e^{2ru}du = A_1(r) + B_1(r)$$

$$= \tfrac{1}{2}\pi^{-\frac{1}{2}r-1/4}(r+5/2)(r+\tfrac{1}{2})\Gamma(\tfrac{1}{2}r+1/4)$$

$$-(3/2)\pi^{-\frac{1}{2}r-1/4}(r+\tfrac{1}{2})\Gamma(\tfrac{1}{2}r+1/4) + o(1)$$

$$= \tfrac{1}{2}\pi^{-\frac{1}{2}r-1/4}(r+\tfrac{1}{2})\Gamma(\tfrac{1}{2}r+1/4)(r-\tfrac{1}{2}) + o(1)$$

$$= \tfrac{1}{2}\pi^{-\frac{1}{2}r-1/4}(r^2-1/4)\Gamma(\tfrac{1}{2}r+1/4) + o(1). \tag{11}$$

But $\Gamma(x) = (2\pi)^{\frac{1}{2}}x^{x-\frac{1}{2}}e^{-x+\mu(x)}$, where $\mu(x) = (12x)^{-1} + O(x^{-2})$, so that a straightforward calculation shows

$$\Gamma(\tfrac{1}{2}r + 1/4) = (2\pi)^{\frac{1}{2}}(\tfrac{1}{2}r)^{\frac{1}{2}r-1/4}e^{-\frac{1}{2}r}(1 - 1/(48r) + O(r^{-2}))$$

Substitution of this result into (11) gives

$$\begin{aligned} I_1 &= \int_0^\infty \Phi_1(u)e^{2ru}du \\ &= \tfrac{1}{2}\pi^{-\frac{1}{2}r-1/4}(r^2 - 1/4)(2\pi)^{\frac{1}{2}}(\tfrac{1}{2}r)^{\frac{1}{2}r}(2/r)^{1/4}e^{-\frac{1}{2}r}(1 - 1/(48r) + O(r^{-2})) \\ &= (\pi/2)^{1/4}r^{\frac{1}{2}r}(2\pi e)^{-\frac{1}{2}r}(r^{7/4} - (4r^{1/4})^{-1}(1 - 1/(48r) + O(r^{-2})) \\ &= (\pi/2)^{1/4}r^{\frac{1}{2}r}(2\pi e)^{-\frac{1}{2}r}(r^{7/4} - r^{3/4}/48 + O(r^{-1/4})), \end{aligned}$$

and the contributions of $\int_0^\infty \Psi_1(u)e^{2ru}du$, *II* may be merged into the O-term, so that the expansion calculated for I_1 serves also for $M(r)$. (Cf. (5), (6)). From this it is seen that the appraisal of $\int_0^\infty \Psi_k(u)e^{2ru}du$ could have been confined to the case $k = 1$.

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SMALL PERTURBATIONS.*

By AUREL WINTNER.

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1. Let $A = A(t)$, where $A = (a_{ik})$ and $i, k = 1, \cdots, n$, be a matrix of n^2 functions $a_{ik} = a_{ik}(t)$ defined and continuous for $0 \leqq t < \infty$. Then the system of differential equations $x' = A(t)x$, where $x = (x_1, \cdots, x_n)$ and $x' = dx/dt$, has a unique solution $x = x(t)$, $0 \leqq t < \infty$, assigned by an arbitrary initial vector $x(0)$. Suppose that $A(t)$ becomes *practically* independent of t when t is large (the meaning of such an assumption can be made precise by a number of inequivalent definitions, two of which will be considered in a moment). Then, if A_0 denotes the constant matrix determined by that asymptotic approximation to $A(t)$, various applications assume that, in reasonable situations specific formulations of which could worry only a pure mathematician, the dissipation effected by $A(t) - A_0$ becomes negligible when $t \to \infty$.

Straightforward examples show that the exclusion of secular terms in the general solution of the approximative system $y' = A_0 y$ certainly is a *necessary* restriction. This restriction, which means the exclusion of non-linear elementary divisors for the constant matrix A_0, will always be assumed in what follows.

Certain *sufficient* conditions are known since the work of Liapounoff and Poincaré on the case of analytic differential equations (linear or not). Their results were subsequently extended by Cotton, Perron and others (for references cf. [3], p. 130). The result of Perron, generalizing Poincaré's theorem concerning the case of rational coefficients, assumes that all n^2 elements of $A(t)$ tend to finite limits as $t \to \infty$ and that, if A_0 denotes the limit matrix, the n characteristic numbers of A_0 have n distinct real parts. Under these assumptions, Perron proves that, *on a logarithmic scale*, $x' = A(t)x$ is asymptotically equivalent to $y' = A_0 y$. This does not mean asymptotic behavior proper, since $f_1(t) \sim f_2(t)$ is a weaker statement than $\log f_1(t) \sim \log f_2(t)$ as $t \to \infty$.

2. In contrast, Cotton considers instances of asymptotic behavior proper but assumes that the Poincaré-Perron requirement $A(t) \to A_0$ is replaced by

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the requirement that the deviation of $A(t)$ from A_0 is absolutely integrable, i. e., that

$$\int^{\infty} |a_{ik}(t) - \alpha_{ik}|\, dt < \infty \tag{1}$$

holds for all n^2 elements of $A = (a_{ik})$ and $A_0 = (\alpha_{ik})$. On the other hand, due to the abandonment of the logarithmically weakened scale, only particular, rather than the general, solutions of $A(t)$ can now be discussed. The details of the proof are given for the case $n = 2$ under the assumption that one of the characteristic numbers of A_0 is positive and the other is negative ([2], pp. 479-487). However, the proof is readily seen to remain valid, in case of an arbitrary n, under the assumption that the characteristic numbers of A_0 have n distinct real parts; so that the *algebraical* assumption is the same as in Poincaré's theorem as generalized by Perron.

Except when $n = 1$, this assumption excludes every case of "stable" type (i. e., every case in which all characteristic numbers of A_0 are purely imaginary), as well as the case of every real A_0 having at least one complex characteristic number. However, Cotton explains ([2], pp. 492-493) that his method applies, if $n = 2$, to the case of a conjugate pair of purely imaginary characteristic numbers also. Actually, it is easy to see from the formal structure of the differential equations that, if $n = 2$, the case of an arbitrary conjugate pair of complex characteristic numbers is reducible to the particular case just mentioned (cf. the Remark at the end of the proof of (*) below).

3. If the n characteristic numbers have distinct real parts, the Poincaré-Perron discussion applies to n linearly independent solutions but assumes the logarithmic scale. It was observed after (1) that the latter assumption is essential, i. e., that only an m-parameter sheaf, where $1 \leqq m < n$, can in general exist in the problem of asymptotic behavior proper.

This fact can be proved by the example in which the system $x' = A(t)x$ is represented by

$$x_1' = 1 \cdot x_1 + 0 \cdot x_2 + 0, \qquad x_2' = 0 \cdot x_1 + 0 \cdot x_2 + x_1/t^2, \tag{2}$$

where $n = 2$ and $(x_1, x_2) = x$. In (2), the characteristic numbers of A_0 are real, and both $a_{ik}(t) \to \alpha_{ik}$ and (1) are satisfied. However, while the approximative system $y' = A_0 y$ belonging to (2) is $y_1' = y_1$, $y_2' = 0$ and has therefore the general solution

$$y_1(t) = c_1 e^t, \qquad y_2(t) = c_2,$$

the general solution of (2) itself is seen to be

$$x_1(t) = c_1 e^t, \qquad x_2(t) = c_2 + c_1 \int_1^t (e^u/u^2)\,du$$

$(t > 0)$. Consequently, there is no proper asymptotic relationship within the complete families of the general solutions, but only within the 1-parameter sheaves $c_1 = 0$. In fact, the complementary 1-parameter sheaves $c_2 = 0$ are such that $y_2(t)$ vanishes identically but $x_2(t)$, being an arbitrary constant multiple of the definite integral in the last formula line, tends to ∞ as $t \to \infty$ (if $c_1 \neq 0$).

Incidentally, since this integral clearly is $o(e^t)$, and therefore of a lower order than the components $y_1(t)$, $x_1(t)$, not even an involvement of the latter components could lead to an asymptotic connection between (x_1, x_2) and (y_1, y_2), except within the 1-parameter sheaf $c_1 = 0$.

Thus it is of particular interest that, if Poincaré's assumption of distinct real parts for the characteristic numbers of A_0 is replaced by its precise opposite, namely, by the assumption of "stable" characteristic numbers, a case of first interest in the applications, then (whether the purely imaginary characteristic numbers, some of which can be 0, are or are not distinct) there exists a *proper* asymptotic relation between *every* solution $x(t)$ of the system $x' = A(t)x$ and a corresponding solution $y(t)$ of the trivial system $y' = A_0 y$ (*even if* $n > 2$). This is the content of (*) below.

4. As mentioned at the very beginning, no sharp connection between the system $x' = A(t)x$ and its stationary approximation $y' = A_0 y$ can be expected if the elementary divisors of A_0 (i. e., of the constant matrix (α_{ik}) for which (1) is assumed) are allowed to be multiple. In fact, the additional assumption that all characteristic numbers of A_0 are of stable type is then of no avail. This is shown by the following example, which proves that the corresponding assumption of (*) below cannot be omitted.

Consider the scalar function

$$u(t) = \exp i(t + \log t)\,; \tag{3}$$

so that $|u| = 1$ for every $t(> 0)$. Thus, if $u' = du/dt$,

$$u' = (i + i/t)u \quad \text{and} \quad u'' = (-1 - 2/t + \text{const.}/t^2)u.$$

Hence, if u'', u', u are multiplied by 1, α, β respectively, it follows by addition that, if α and β are so chosen as to satisfy the pair of conditions

$$-1+i\alpha+\beta=0, \qquad -2+i\alpha=0,$$

that is, if $\alpha=-2i$ and $\beta=-1$, then (3) is a solution of the scalar differential equation

$$u''-2iu'+(-1+f)u=0, \tag{4}$$

where $f=f(t)$ is $1/t^2$ times a constant. If (4) is written in the form $x'=A(t)x$, where $x=(x_1, x_2)$ and $x_1=u$, $x_2=u'$, then the assumption (1) is satisfied, since $f(t)=O(1/t^2)$, and what corresponds to $y'=A_0y$ is the scalar differential equation

$$w''-2iw'-w=0. \tag{5}$$

Consequently, if the case of multiple elementary divisors were admissible, then every solution of (4) ought to be asymptotic to an appropriate solution of (5). And this would imply the existence of two constants c_1, c_2 satisfying

$$\exp i(t+\log t)\sim(c_1+c_2t)\exp it$$

as $t\to\infty$. In fact, the function on the right of this asymptotic relation is the general solution of (5), and the function on the left is the solution (3) of (4). But the assertion of the last formula line is equivalent to

$$\exp(i\log t)\sim c_1+c_2t, \qquad t\to\infty,$$

and is therefore false for every pair of constants c_1, c_2.

5. In the particular case $x_2'=x_1$, in which the system $x'=A(t)x$ of $n=2$ linear differential equations of first order for $x=(x_1, x_2)$ is identical with one linear differential equation of second order for x_2, the content of (*) below is classical from that deduction of the asymptotic formula of Bessel's functions $J_m(t)$ which, avoiding the representation of $J_m(t)$ as a Fourier integral, is based directly on the differential equation of J_m (cf., e. g., Picard's indications in [4], pp. 417-418). However, the general case of two equations of first order cannot be reduced to one equation of second order. It is true that such a reduction is possible in a formal fashion. However, the eliminations involved assume that every coefficient function $a_{ik}(t)$ of $x'=A(t)x$, instead of being just continuous, is differentiable (and must, therefore, be required to have a continuous derivative). But even if every $a_{ik}(t)$ is restricted to be analytical, it is not clear that the fundamental assumption corresponding to (1) is satisfied for the resulting differential equation of the second order, when (1) is satisfied for the given pair of differential equations of the first order.

However, the following theorem will be proved for *every* n.

(*) *Let* $A(t) = (a_{ik})$, *where* $0 \leqq t < \infty$ *and* $i, k = 1, \cdots, n$, *be a matrix of* n^2 *continuous functions* $a_{ik} = a_{ik}(t)$ *corresponding to which there exists a matrix* $A_0 = (\alpha_{ik})$ *of* n^2 *constants* α_{ik} *satisfying* (1). *Suppose that the general solution of the differential equations* $y' = A_0 y$, *where* $y = (y_1, \cdots, y_n)$, *is a superposition of simple vibrations (i. e., that all elementary divisors belonging to* A_0 *are linear and that all characteristic numbers of* A_0, *which need not be distinct, have a common real part, which is* 0). *Then every solution* $x(t)$ *of the system* $x' = A(t)x$, *where* $x = (x_1, \cdots, x_n)$, *determines a finite almost-periodic sum* $y(t)$ *which satisfies* $x(t) - y(t) \to 0$ *as* $t \to \infty$ *and is a solution of the trivial system* $y' = A_0 y$.

Aside from the circumstance that (*) takes care of all the "stable" cases of the applications, the interest of (*) is three-fold: The asymptotic statement of (*), in contrast to that of the Poincaré-Perron theorem, refers to the *proper*, rather than to the logarithmically weakened scale and supplies, in contrast to the possibility existing in the example (2) belonging to $n = 2$, the ultimate behavior of *all* solutions of $x' = A(t)x$, although it allows *every* n.

6. The classical case of a linear differential equation of the second order which "nearly" is the equation $u'' + \lambda^2 u = 0$ of a linear oscillator represents a particular case of (*). In fact, (*) implies the following

COROLLARY. *If* $f = f(t)$, $0 \leqq t < \infty$, *is a continuous function satisfying*

$$\int^{\infty} |f(t)|\, dt < \infty, \tag{6}$$

and if λ *is a positive number, then every solution* $u(t)$ *of the differential equation*

$$u'' + (\lambda^2 + f)u = 0 \tag{7}$$

determines two constants a, b *for which*

$$u(t) - (a \cos \lambda t + b \sin \lambda t) \to 0 \tag{8}$$

holds as $t \to \infty$. *(In addition, the differentiation of this limit relation is legitimate, that is,*

$$u'(t) - (\alpha \cos \lambda t + \beta \sin \lambda t) \to 0, \tag{8 bis}$$

where $\alpha = \lambda b$, $\beta = -\lambda a$.)

In order to see this, it is sufficient to observe that, if the scalar equation (7), where $f = f(t)$, is identified with $x' = A(t)x$, and the approximation $w'' + \lambda^2 w = 0$ to (7) with $y' = A_0 y$, then the case $n = 2$ of (*) is applicable,

since the assumption (1) appears in the form (6) and the constant matrix A_0 becomes

$$\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix};$$

a matrix having the distinct, purely imaginary characteristic numbers $\pm \lambda i$.

It is instructive to illustrate the content of the Corollary by applying it to the differential equation,

$$v'' + v'/t + (1 - m^2/t^2)v = 0, \tag{9}$$

of Bessel's function $J_m(t)$. If $u = t^{\frac{1}{2}}v$, then (9) appears in the self-adjoint form (7), where $\lambda = 1$ and $4f = (1 - 4m^2)/t^2$. Hence, $f(t) = O(1/t^2)$ as $t \to \infty$. Since this implies that (6) is satisfied, (8) is applicable and supplies, since $\lambda = 1$ and $v = u/t^{\frac{1}{2}}$, the existence of two constants a, b for which the asymptotic formula

$$v(t) = (a \cos t + b \sin t)/t^{\frac{1}{2}} + o(1/t^{\frac{1}{2}}) \tag{10}$$

holds as $t \to \infty$. This agrees with the standard relation

$$J_m(t) = (\tfrac{1}{2}\pi t)^{-\frac{1}{2}} \cos(t - \tfrac{1}{2}(m + \tfrac{1}{2})\pi) + O(1/t). \tag{10 bis}$$

Actually, a comparison of (10) with (10 bis) clears up the methodical situation in two different respects:

(α) The constants a, b, for which (10 bis) supplies the values

$$a = (\tfrac{1}{2}\pi)^{-\frac{1}{2}} \cos \tfrac{1}{2}(m + \tfrac{1}{2})\pi, \qquad b = (\tfrac{1}{2}\pi)^{-\frac{1}{2}} \sin \tfrac{1}{2}(m + \tfrac{1}{2})\pi,$$

are left undetermined by the result (10). This is quite natural, since a, b in (10) are arbitrary integration constants. In fact, (10) holds not only for the particular solution $v = J_m(t)$ of (9) but for any $v = \text{const.}\, J_m(t)$ as well and, in addition, for those solutions $v(t)$ of (9) which contain the solution linearly independent of $J_m(t)$.

(β) The remainder term of (10 bis) is sharper than that of (10). This is quite natural, since, on the one hand, (10) is identical with (8) in virtue of $u = t^{\frac{1}{2}}v$ and, on the other hand, (8) is a result depending only on the general assumption (6), rather than on the drastic estimate $f(t) = O(1/t^2)$, or even on the explicit relation $f(t) = \text{Const.}/t^2$, available in the present particular case. Actually, if the assumption (6) is refined to $f(t) = O(1/t^2)$, then the $o(1/t^{\frac{1}{2}})$ in (10) can be replaced by $O(1/t)$, since the remainder term of (8), which is $o(1)$, can then be improved to $O(1/t^{\frac{1}{2}})$. The truth of this

assertion, and also of its extension to the case of similar explicit assumptions (such as $f(t) = O(1/t^{1+\theta})$, where $\theta > 0$) will become clear from the proof of the general result (*), of which (8) was a corollary.

7. What concerns (*) itself, the substance of its assertion is a simple "Abelian" (in contrast to some "Tauberian") lemma. The lemma in question is that particular case of (*) which results when A_0 is chosen to be the matrix (0); a matrix which satisfies the requirements of (*), since it is a diagonal matrix in which all diagonal elements are purely imaginary, namely, 0. But if $A_0 = (0)$, then every solution $y(t)$ of $y' = A_0 y$ is independent of t. Consequently, if α_{ik}, $a_{ik}(t)$, x in (*) are replaced by 0, $f_{ik}(t)$, z respectively, there results the following fact:

(i) *If $F(t) = (f_{ik})$, where $0 \leqq t < \infty$ and $i, k = 1, \cdots, n$, is a matrix of n^2 continuous functions $f_{ik} = f_{ik}(t)$ satisfying*

$$\int^{\infty} |f_{ik}(t)|\, dt < \infty, \tag{11}$$

then each of the components $z_i = z_i(t)$ of every solution $z = z(t)$ of the system $z' = F(t)z$, where $z' = dz/dt$ and $z = (z_1, \cdots, z_n)$, tends to a finite limit as $t \to \infty$.

Actually, it will turn out that this primitive Abelian theorem contains (*) as a corollary.

It is a general principle that an Abelian theorem, which concerns a limiting equality, represents the extreme case of a less sharp but more fundamental fact, which supplies limiting inequalities (involving, if possible, absolute constants). However, (i) can hardly be improved in this direction, since, on the one hand, the assertion of (i) proves to be equivalent to the much rougher assertion

$$z(t) = O(1), \qquad (t \to \infty), \tag{12}$$

and, on the other hand, nothing less than the assumption (11) can insure that (12) holds for every solution $z(t)$ of $z' = F(t)z$ (cf. the proof of (i) below).

Consequently, all that it is reasonable to ask for is that deteriorated form of the estimate (12) which results if the assumption (11) is lightened somewhat. The simplest specific order not implying (11) is the order of $o(1/t)$ for every $f_{ik}(t)$. But all that can be said in this case is that $z(t) = O(t^{\epsilon})$ holds for every fixed $\epsilon > 0$ (rather than, as in the case (12) of (11), for $\epsilon = 0$). For

the sake of possible reference in the proofs to be given below, let the positive part of this assertion be worded as a formal remark:

(i bis) *If $F(t)$ is a matrix of n^2 continuous functions which are $o(1/t)$ as $t \to \infty$, then each of the n components of every solution $z(t)$ of $z' = F(t)z$ is $O(t^\epsilon)$ for every fixed $\epsilon > 0$.*

Actually, (i bis) will turn out to be just a limiting case of (ii) below.

8. The "Abelian" nature of the above assertions suggests that some of the fundamental theorems in the classical theory of linear differential equations are disguised formulations of more general and quite primitive facts, which can be completely isolated from the theory of analytical functions of a complex variable. Such an approach can be expected to furnish not only more *general* theorems than, but also *simplified* forms for the proofs of, the classical results, since the arrangement must be so primitive as to hold under the mere assumption of continuous functions of a real variable, and to be able to retard the introduction of the specific complex-analytical restriction.

In case of Fuchs' fundamental theorem concerning "regular" singular points, the possibility of such an isolation of the underlying general facts, which have nothing to do with analyticity, is indicated by Schlesinger's proof of this theorem, as given in [5], pp. 168-170 (cf. also the presentation in [6], pp. 142-145, where reference is given to a paper of Fuhr). In this regard, it is instructive to consider Bieberbach's reproduction ([1], pp. 175-176) of Birkhoff's variant of Schlesinger's proof. In fact, the proof as presented by Bieberbach is quite incomplete, since the least attractive part of the necessary considerations, and even an observation as to the existence of such a part, is left to the ingenuity of that legendary beginner for whom text-books are written. The part in question ought to concern the differentiability of the squared absolute value of the functions involved, which are regular. This differentiability property *can*, of course, be proved. That it *needs* a proof, is clear from the fact that the absolute values themselves are not differentiable at simple zeros. *If* the latter cluster at the singular point, the proof ceases to be one. And the zeros *could* cluster, since the fact that the singular point does not correspond to an essential singularity of the solutions is just the statement to be proved and cannot therefore be used in the proof. Thus the presentation of Fuchs' theorem in [1] affords an instance of the disadvantages which can arise if facts depending on analyticity are not isolated from those having nothing to do with it.

Schlesinger's proof, as well as its subsequent variants, depends on his

observation that, instead of all the lengthy formal work involved in the earlier proofs, nothing but a uniform estimate of finite order is needed for the solutions (the uniformity refers to all directions issuing from the singular point in the complex plane). But this in itself suggests the possibility of isolating a general fact of the type described above. This general fact is the content of the following theorem, the proof of which will be just as primitive as that of (i). Not even an application of Liapounoff's square-sum device (cf., e. g., [4], pp. 382-385), on the use of which Birkhoff's variant of Schlesinger's proof depends, is needed.

(ii) *If $F(t)$ is a matrix of n^2 continuous functions of the real variable t all of which are $O(1/t)$ as $t \to \infty$, then each of the n components of any solution $z(t)$ of the system $z' = F(t)z$ is $O(t^c)$ as $t \to \infty$, where c is a sufficiently large constant, depending only on the values of the n^2 constants which multiply $1/t$ in $O(1/t)$.*

In fact, the proof of (ii) will be such as to show that c can be chosen to be any number greater than the absolutely greatest characteristic number of the matrix (β_{ik}) defined by

$$\beta_{ik} = \limsup_{t \to \infty} |f_{ik}(t)|/t,$$

where $(f_{ik}) = F$ (according to Frobenius, this characteristic number is real and non-negative, since $\beta_{ik} \geqq 0$). In particular, if the assumption of (ii) is replaced by that of (i bis), then, since (β_{ik}) becomes the matrix (0), the constant c can be chosen to be any $\epsilon > 0$, as stated by (i bis).

Of the assertions italicized above, (*), (i), (ii) remain to be proved. It is logical to prove these three assertions in the order reverse to that in which they were stated, since their degree of coarseness is monotone decreasing when (*) is at the first place.

9. Proof of (ii). By assumption, there exists a constant matrix B satisfying the n^2 inequalities $|f_{ik}(t)| \leqq b_{ik}/t$, where $(f_{ik}) = F$, $(b_{ik}) = B$ and, without loss of generality, $1 \leqq t < \infty$. With reference to any fixed solution $z(t) = (z_1, \cdots, z_n)$ of the system $z' = F(t)z$, let $s(t) = (s_1, \cdots, s_n)$ denote that solution of the system $s' = Bs/t$ for which the n initial values $s_i(1)$ are the absolute values of the corresponding components of the initial vector $z(1)$. Let $z^m(t)$ and $s^m(t)$ denote, respectively, the infinite sequences of functions defined by the successive approximations

$$z^m(t) = z(1) + \int_1^t f(u) z^{m-1}(u)\, du \quad \text{and} \quad s^m(t) = s(1) + \int_1^t B s^{m-1}(u)/u\, du,$$

where $z^1(t) = z(1)$ and $s^1(t) = s(1)$ for every $t(\geqq 1)$. Then it is clear from the definitions of B and $s_i(1)$ that $|z_i^m(t)| \leqq s_i^m(t)$ holds for every t and for every positive integer m (the subscript i refers to any of the n components of the vectors z^m, s^m).

Since both systems of differential equations are linear, so that the successive approximations $z^m(t,)$, $s^m(t)$ converge, as $m \to \infty$, to the corresponding solutions $z(t)$, $s(t)$ for every t, it follows from the last inequality that $|z_i(t)| \leqq s_i(t)$ holds for every t. Consequently, the assertion of (ii) will be proved if it is ascertained that each of the n components of every solution $s(t)$ of the system $s' = Bs/t$ is $O(t^c)$ as $t \to \infty$, where c is a constant depending on B.

The substitution $t = \exp t^*$ transforms the system $s' = Bs/t$, where $s' = ds/dt$, into the system $ds/dt^* = Bs$, which has constant coefficients (Euler-Cauchy). Hence, the general solution of the latter system is a linear combination of $\exp(\mu_1 t^*), \cdots, \exp(\mu_n t^*)$, if the characteristic numbers $\mu_1, \cdots, \mu_n$ of B are all distinct. If they are not distinct, there is the possibility (though not the necessity) that some of the exponentials acquire the factors $\log t^*$, $\log^2 t^*, \cdots, \log^{n-1} t^*$, each of which is $O(t^*)^\epsilon$ for every fixed $\epsilon > 0$. In any case, if μ denotes the greatest of the real parts of the n characteristic numbers μ_i, every solution s is seen to be $O \exp(ct^*)$, where $c = \mu + \epsilon$ and $\epsilon > 0$. Since $t^* = \log t$, this O-estimate completes the proof of (ii).

10. Suppose that the $O(1/t)$ of the hypothesis of (ii) is generalized to $O(1/t^\mu)$, where μ is a fixed positive index. In contrast to the "rank" in the Poincaré-Perron-Birkhoff theory of Thomé's normal series for the analytical case, μ need not be an integer. But this contrast is unessential, since the theory of "rank" can be transcribed to the so-called hypercontinuous case, in which case the coefficient functions, instead of being analytical, have derivatives of arbitrarily high order and possess formal expansions proceeding according to powers of t, which can be fractional. What *is* essential is that no such expansions need now exist for the functions occurring in the coefficient matrix, which are allowed to be arbitrary continuous functions of t.

It is clear from the proof of (ii), where $\mu = 1$, that the estimate supplied by the assertion of (ii) can in no sense be improved. On the other hand, if $\mu > 1$, then (i) is applicable, and so the estimate in the limiting case $\mu = 1$ of (ii) appears as being quite rough. However, if $\mu = 1$ is thought of as the limiting case of the case $\mu < 1$, then the estimate supplied by the assertion of (ii) becomes sharp indeed, since all that can in general be true in the case $\mu < 1$ is contained in the following fact:

(ii bis) *If $F(t)$ is a matrix of n^2 continuous functions which, as $t \to \infty$, are $O(1/t^\mu)$, where $0 < \mu < 1$, then each of the n components of every solution of the system $z' = F(t)z$ is $O(e^{ct^{1-\mu}})$ as $t \to \infty$, where c is a sufficiently large constant, depending only on μ and on the n^2 constants multiplying $1/t^\mu$ in $O(1/t^\mu)$.*

In fact, the "best" majorant system for the successive approximations to the solution of $z' = F(t)z$ is now a system of the form $s' = Bs/t^\mu$, where the notations are the same as in the above proof of (ii); so that B is a real constant matrix with non-negative elements, and s is a vector $(s_1, \cdots, s_n)$. But s' denotes ds/dt, and so it is readily seen that the introduction of the (generalized) logarithmic time-scale, defined by the substitution

$$t \to (\log t)/(1-\mu), \qquad (0 < \mu < 1),$$

transforms the majorant system $s' = Bs/t^\mu$ of $z' = F(t)z$ into $s' = Cs/t$, where the last prime refers to differentiation with respect to the new t and C denotes $1/(1-\mu)$ times the constant matrix B. (The factor $1/(1-\mu)$, occurring in the definitions of the t-transformation and of C, is positive, since $0 < \mu < 1$.) But if c is any number greater than the real part of any of the n characteristic numbers of the constant matrix C, then, for that trivial reason on which the end of the above proof of (ii) depends, each of the n components of every solution $s = s(t)$ of $s' = Cs/t$ is $O(t^c)$ as $t \to \infty$.

Since the inverse of the substitution of the last formula line is the transformation

$$t \to e^{t^{1-\mu}}, \qquad (0 < \mu < 1),$$

and since the latter transforms the system $s' = Cs/t$ into the original majorant $s' = Bs/t^\mu$ of the given system $z' = F(t)z$, the estimate $O(t^c)$ of the solutions of $s' = Cs/t$ proves the exponential estimate of the assertion of (ii bis). It is also seen that this exponential estimate cannot be improved, since $F(t)$ may be exactly $1/t^\mu$ times the constant matrix B.

11. Proof of (i). Let any given solution $x(t) = (x_1, \cdots, x_n)$ of $x' = F(t)x$, where $F(t) = (f_{ik})$, be thought of as determined by the initial vector $x(c)$, where c is a fixed value of t. According to (11), it is possible to choose c so large that

$$\sum_{k=1}^{n} \int_c^\infty |f_{ik}(t)|\, dt \leqq \tfrac{1}{2} \tag{13}$$

holds for all n values of i. It will be assumed that c is fixed in this manner, and that $c \leqq t < \infty$.

Let C_m, where $m = 1, 2, \cdots$ (and, for the present, $C_m \leqq \infty$) be the least common upper bound of the n functions

$$|z_1^{m+1}(t) - z_1^m(t)|, \cdots, |z_n^{m+1}(t) - z_n^m(t)| \tag{14}$$

for $c \leqq t < \infty$, where, if $z_1(c), \cdots, z_n(c)$ are the initial constants determining a solution $z_1(t), \cdots, z_n(t)$, the functions $z_1^0(t), \cdots, z_n^0(t)$ are respectively equal to $z_1(c), \cdots, z_n(c)$ for every t and the remaining functions $z_i^m(t)$ are those assigned by the recursion formula of the successive approximations, that is, by

$$z_i^{m+1}(t) = z_i(c) + \int_c^t \sum_{k=1}^n f_{ik}(u) z_k^m(u)\, du.$$

If m is here replaced by $m - 1$, it follows, by subtraction, that the absolute value of the i-th of the n functions (14) does not exceed

$$\int_c^t \sum_{k=1}^n |f_{ik}(u)|\ |z_k^m(u) - z_k^{m-1}(u)|\, du \leqq \int_c^\infty \sum_{k=1}^n |f_{ik}(u)|\, C_{m-1}\, du;$$

a bound which, according to (13), is majorized by $\frac{1}{2}C_{m-1}$. Consequently, $C_m \leqq \frac{1}{2}C_{m-1}$. Hence, $C_1 + C_2 \cdots$ is a convergent series. Since C_m majorizes all n functions (14) for every t, and since the n functions $z_i^1(t)$ are independent of t, it follows that the functions $z_1^m(t), \cdots, z_n^m(t)$, where $m = 1, 2, \cdots$ and $c \leqq t < \infty$, are uniformly bounded. This proves (12), since $z^m(t) \to z(t)$ as $m \to \infty$ holds for every t.

Although (12) claims less than (i), it implies (i) in virtue of (13). In fact, since $z(t)$ is a solution of $z' = F(t)z$,

$$z(t) = z(c) + \int_c^t F(u)z(u)\, du.$$

Hence, the assertion of (i) is equivalent to the convergence of the integral

$$\int_c^\infty F(t)z(t)\, dt.$$

But (12) and (11), where $(f_{ik}) = F$, imply the *absolute* convergence of this integral; so that the proof of (i) is complete.

12. Proof of (*). The assumptions and the assertion of (*) are invariant under transformations of x into Tx, where T denotes a constant

matrix of non-vanishing determinant. Since the elementary divisors belonging to A_0 are supposed to be linear, it follows that A_0 can be assumed to be a diagonal matrix. Then, if $\lambda_1, \cdots, \lambda_n$ denote the characteristic numbers of A_0, the system $x' = A(t)x$ appears in the form

$$x_i' = \lambda_i x_i + \sum_{k=1}^{n} f_{ik}(t) x_k, \qquad (i = 1, \cdots, n), \tag{14}$$

where $f_{ik}(t)$ represents the continuous function which, in the notations of the wording of (*), is the difference $a_{ik}(t) - \alpha_{ik}$. Hence, the assumption (1) of (*) means that (11) is satisfied by (14).

The trivial system $y' = A_0 y$ belonging to the representation (14) of $x' = A(t)x$ is the system $y_i' = \lambda_i y_i$ and has, therefore, the general solution $y_i(t) = c_i e^{\lambda_i t}$, where $i = 1, \cdots, n$. Accordingly, the assertion of (*) is equivalent to the statement that, if $(x_1, \cdots, x_n)$ is any solution of (14), each of the n functions

$$z_i(t) = e^{-\lambda_i t} x_i(t) \tag{16}$$

tends to a finite limit, c_i, as $t \to \infty$. Consequently, it is sufficient to show that the n functions (16) represent a solution of some system $z' = F(t)z$ to which (i) is applicable. But such a system results if (14) is expressed in terms of the z_i defined by (16).

In fact, if $x_i = e^{\lambda_i t} z_i$ and its derivative x_i', which is identical with $e^{\lambda_i t}$ times $z_i' + \lambda_i z_i$, are substituted into (14), the resulting transform of (14) may be written in the form

$$z_i' = \sum_{k=1}^{n} e^{(\lambda_k - \lambda_i)t} f_{ik}(t) z_k. \tag{17}$$

This system is of the type $z' = F(t)z$, except that the matrix element f_{ik} of $F = (f_{ik})$, instead of being the function $f_{ik}(t)$ occurring in (14), is $e^{(\lambda_k - \lambda_i)t}$ times that $f_{ik}(t)$. But the exponential function multiplying the latter $f_{ik}(t)$ is of absolute value 1, in virtue of that assumption of (*) which has not been used so far, namely, of the assumption that $\lambda_1, \cdots, \lambda_n$ are purely imaginary (possibly 0). Since (1) is satisfied by the functions $f_{ik}(t)$ occurring in (17), it follows that (1) is satisfied by the elements $f_{ik}(t)$ of the matrix $F = (f_{ik})$ also. This proves that (i) is applicable to the representation $z' = F(t)z$ of (17).

Remark. What is actually used in this proof of (*) is, not the full force of the assumption that all n characteristic numbers λ_i be purely imaginary, but merely the assumption that the n^2 exponential factors be of absolute value

1 for every t. And this will be the case whenever all n^2 differences $\lambda_i - \lambda_k$ are purely imaginary (including 0). But all that this assumption requires is an arbitrary common real part for all characteristic numbers. Thus the true content, though not the almost-periodic wording, of (*) actually is a theorem, the algebraical assumption of which is just the opposite of the corresponding assumption of Poincaré's theorem (in which the real parts of the characteristic numbers are required to be all distinct).

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SPANS IN LEBESGUE AND UNIFORM SPACES OF TRANSLATIONS OF PEAK FUNCTIONS.*

By RALPH PALMER AGNEW.

1. Introduction. For each $p \geqq 1$, let L_p be the Lebesgue space whose elements are the real or complex valued measurable functions $f(x)$ such that $|f(x)|^p$ is integrable over $-\infty < x < \infty$. The distance $\| f_2 - f_1 \|$ between two elements f_1 and f_2 of L_p is defined by

$$\| f_2 - f_1 \| = \{ \int_{-\infty}^{\infty} | f_2(x) - f_1(x) |^p \, dx \}^{1/p}. \tag{1.1}$$

Let E be a set in L_p. The linear manifold $M(E)$ determined by E is the set of all linear combinations (finite) of elements of E. The *span* $S_p(E)$ of E in L_p is the closure in L_p of $M(E)$.

Let $F \in L_p$. For each real h, the translation $F(x+h)$ of $F(x)$ is also in L_p. Let T_F denote the set of translations of F. Wiener [4] showed that, when $F \in L_1$, $S_1(T_F) = L_1$ if, and only if, the Fourier transform of F has no real zeros; and that, when $F \in L_2$, $S_2(T_F) = L_2$ if, and only if, the set of zeros of the Fourier transform of F has measure 0. Segal [3] complicated matters by showing that, when $1 < p < 2$, there is an element F of L_p such that (i) the zeros of the Fourier transform of F have measure 0 and (ii) the span $S_p(T_F)$ of the translations of F does *not* include the whole space L_p. The author [1] has shown that if $p > 1$ and F is a simple step function, then $S_p(T_F) = L_p$.

It is the main object of this paper to prove the following theorem by a direct method which shows how one may approximate a given function in L_p by linear combinations of translations of a given peak function.

THEOREM 1.2. *If $p > 1$ and F is a peak function, then $S_p(T_F) = L_p$.*

Uniform approximation and truncated peak functions are treated in **6** and **7**.

2. Peak functions. In terms of positive constants a and b, a *peak function* $F(x)$ of *width* $2b$ and *height* ab is defined by the formula

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$$(2.1) \qquad \begin{aligned} F(x) &= a(b - |x|) && |x| \leqq b \\ &= 0 && |x| > b. \end{aligned}$$

The Fourier transform $G(u)$ of this function $F(x)$ is the continuous non-negative function for which $G(0) = ab^2(2\pi)^{-1/2}$ and

$$(2.2) \qquad G(u) = (1/2\pi)^{1/2} \int_{-\infty}^{\infty} F(x) e^{iux} dx = (2/\pi)^{1/2} a \frac{1 - \cos bu}{u^2}$$

when $u \neq 0$. Wiener's results imply that $S_1(T_F) \neq L_1$ and $S_2(T_F) = L_2$.

3. A lemma. In our proof of Theorem 1. 2, we shall use the following lemma three times.

LEMMA 3. 1. *Let $p > 1$. Let $f(x)$ be a bounded measurable function vanishing outside the interval $c - d < x < c + d$, and let $f(x + d) = f(x)$ [or $f(x + d) = -f(x)$] when $c - d \leqq x \leqq c$. Let $g(x) = f(x)$ when $c - d \leqq x \leqq c$ and $g(x) = 0$ otherwise. Then $S_p(T_f)$ contains g.*

Let $\epsilon > 0$. We establish the lemma by giving an explicit method for selecting a linear combination f_ϵ of translations of f such that $\| g - f_\epsilon \| < \epsilon$. Let M be a constant such that $|f(x)| \leqq M$ for each x. Let n be the least positive integer such that

$$(3.2) \qquad M d^{1/p} (n + 1)^{1/p} n^{-1} < \epsilon.$$

Let

$$(3.3) \qquad f_\epsilon(x) = f(x) + \sum_{k=1}^{n} (-1)^{k+\delta} (1 - k/n) f(x - kd)$$

where $\delta = 0$ if $f(x + d) = f(x)$ and $\delta = 1$ if $f(x + d) = -f(x)$. Then $|g(x) - f_\epsilon(x)|$ is zero except over $n + 1$ intervals, each of length d, where it is at most M/n. Therefore

$$(3.4) \qquad \| g - f_\epsilon \| \leqq \{ \int_0^{(n+1)d} (M/n)^p dx \}^{1/p} = M d^{1/p} (n + 1)^{1/p} n^{-1} < \epsilon$$

and the lemma is proved.

4. Approximation to narrower peak functions. We shall show how to approximate narrower peak functions by linear combinations of translations of a given one by proving the following lemma.

LEMMA 4.1. *Let $p > 1$. Let $F(x)$ be the peak function of width $2b$ defined by (2.1), and let $F_1(x)$ be the narrower peak function, of width $b/2$, defined by*

$$F_1(x) = a(b/4 - |x|) \qquad |x| \leqq b/4$$
$$= 0 \qquad |x| > b/4. \tag{4.2}$$

Then $S_p(T_F)$ contains F_1.

Let $b_1 = b/4$ and let

$$f_1(x) = F(x + 2b_1) + F(x - 2b_1) - F(x). \tag{4.3}$$

Let $g_1(x)$ be the function defined by

$$\begin{aligned} g_1(x) &= a(x + 6b_1) && -6b_1 \leqq x \leqq -4b_1 \\ &= 2ab_1 && -4b_1 \leqq x \leqq -2b_1 \\ &= -ax && -2b_1 \leqq x \leqq 0 \\ &= 0 && \text{otherwise.} \end{aligned} \tag{4.4}$$

The hypotheses of Lemma 3.1 are satisfied when $f = f_1$, $g = g_1$, $c = 0$, and $d = 6b_1$. Hence the span of the translations of f_1 includes g_1 and therefore the span of the translations of F includes g_1.

Let

$$f_2(x) = F(x) - 2g_1(x - 3b_1) \tag{4.5}$$

and

$$\begin{aligned} g_2(x) &= a(x + 4b_1) && -4b_1 \leqq x \leqq -3b_1 \\ &= -a(x + 2b_1) && -3b_1 \leqq x \leqq -b_1 \\ &= ax && -b_1 \leqq x \leqq 0 \\ &= 0 && \text{otherwise.} \end{aligned} \tag{4.6}$$

The hypotheses of Lemma 3.1 are satisfied when $f = f_2$, $g = g_2$, $c = 0$ and $d = 4b_1$. Hence the span of the translations of f_2 includes g_2 and therefore the span of the translations of F includes g_2.

Finally, the hypotheses of Lemma 3.1 are satisfied when $f(x) = g_2(x)$

and $g(x) = F_1(x + 3b_1)$. Hence the span of the translations of g_2 includes $F_1(x + 3b_1)$; therefore the span of the translations of F includes $F_1(x + 3b_1)$ and hence also $F_1(x)$. This completes the proof of Lemma 4.1.

By repeated application of Lemma 4.1, and use of the fact that if $S_p(T_f)$ contains g then $S_p(T_f)$ contains $S_p(T_g)$, we obtain the following lemma.

LEMMA 4.7. *If F is a peak function of width $2b$ and G is a peak function of width*

$$\delta = 2b/4^m \tag{4.8}$$

where m is a positive integer, then $S_p(T_F)$ contains G.

5. Proof of Theorem 1.2. Let $p > 1$, let $f \in L_p$, and let F be a given peak function defined by (2.1). Let $\epsilon > 0$. By a fundamental theorem on approximation, there is a continuous function $f_1(x)$, vanishing outside some finite interval $-\alpha \leqq x \leqq \alpha$, such that $\| f - f_1 \| < \epsilon/3$. We may choose a positive number δ_1 such that the number δ defined by $\delta = 2\delta_1$ has the form (4.8), and moreover the number δ_1 is so small that $\| f_2 - f_1 \| < \epsilon/3$ when $f_2(x)$ is the polygonal function for which $f_2(n\delta_1) = f_1(n\delta_1)$ when $n = 0, \pm 1, \pm 2, \cdots$ and which is linear over the interval $n\delta_1 \leqq x \leqq (n+1)\delta_1$ for each $n = 0, \pm 1, \pm 2, \cdots$. The polygonal function is a finite linear combination of peak functions of width $\delta = 2\delta_1$; in fact if

$$\begin{aligned} G(x) &= \delta_1 - |x| && |x| \leqq \delta_1 \\ &= 0 && |x| > \delta_1, \end{aligned} \tag{5.1}$$

then

$$f_2(x) = \sum f_2(n\delta_1)\, G(x - n\delta_1), \tag{5.2}$$

the sum ranging over the finite set of values of n for which $f_2(n\delta_1) \neq 0$. Since Lemma 4.7 implies that $S_p(T_F)$ contains $G(x)$, it follows from (5.2) that $S_p(T_F)$ contains $f_2(x)$. Hence there is a linear combination $F_\epsilon(x)$ of translations of $F(x)$ such that $\| f_2 - F_\epsilon \| < \epsilon/3$. Our inequalities imply that $\| f - F_\epsilon \| < \epsilon$. Hence $S_p(T_F)$ contains f and Theorem 1.2 is proved.

6. Uniform approximation. A modification of the above proof of Theorem 1.2 furnishes a proof of the following theorem on uniform approximation.

THEOREM 6.1. *Let $F(x)$ be a peak function defined by (2.1). Let $f(x)$ be a continuous function for which $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$. Then to each $\epsilon > 0$ corresponds a linear combination $F_\epsilon(x)$ of translations of $F(x)$ such that*

$$|f(x) - F_\epsilon(x)| < \epsilon \qquad -\infty < x < \infty.$$

The author is indebted to the referee who pointed out to him that this theorem is implied by results of Boas and Bochner [2]. In the latter paper, the italicized statement on page 287 deals with functions $f(t)$, $g(t)$, and a weight function $p(t)$. If we assume that $p(t) \equiv 1$ and that $g(t) \in L$, then g has an ordinary Fourier transform and the approximation

$$|f(t) - \sum_{j=1}^{k} A_j g(t + \lambda_j)| < \epsilon$$

is possible whenever (i) $f(t)$ and $g(t)$ are continuous, except for discontinuities of the first kind in $g(t)$, and have limits 0 for $x \to \pm\infty$ and (ii) there is no interval over which the Fourier transform of $g(t)$ vanishes everywhere.

7. Truncated peak functions. Lemma 3.1 and Theorem 1.2 provide a basis for a short proof of the following theorem.

THEOREM 7.1. *Let $p > 1$. Let $\phi(x)$ be the truncated peak function defined in terms of positive constants a, b, and q for which $q < b$ by the formula*

$$\begin{aligned} \phi(x) &= a(b - q) && |x| \leqq q \\ &= a(b - |x|) && q \leqq |x| \leqq b \\ &= 0 && |x| \geqq b. \end{aligned} \tag{7.2}$$

Then $S_p(T_\phi) = L_p$.

To prove this, let

$$\phi_1(x) = \phi(x) + \phi(x + b + q) \tag{7.3}$$

and

$$f(x) = \phi_1(x) - \phi_1(x - b + q). \tag{7.4}$$

Then, when $g(x)$ is a translation of the peak function (2.1), the hypotheses of Lemma 3.1 hold. Hence, by Lemma 3.1, $S_p(T_f)$ includes g and therefore

$S_p(T_\phi)$ includes g. Since, by Theorem 1.2, $S_p(T_g) = L_p$ it follows that $S_p(T_\phi) = L_p$ and Theorem 7.1 is proved.

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SOLUTION OF A PROBLEM OF G. KOETHE.*

By JAKOB LEVITZKI.

1. Preliminaries. The following problem has been proposed by G. Koethe:[1] Is it possible to deduce the nilpotency of a nil-ring by assuming the maximal condition (in short: M-condition)[2] for the r. ideals? This problem has been generalized by R. Baer, who says [1, p. 553] "it seems to be an open question whether or not the maximum condition for r. ideals is sufficient for nilpotence of the upper radical." In the present note[3] we answer the question of Baer, and by implication also that of Koethe, in the affirmative. Moreover, since it is not known whether, in general, the upper radical contains all one-sided nil-ideals of the ring, it is of interest to note that the M-condition for the r. ideals implies the nilpotency of all one-sided nil-ideals of the ring (compare Theorem 5 of the present note) and thus ensures the existence and the nilpotency of the radical in the sense of Koethe [3, p. 169]. An analogous result concerning the minimal condition (in short: m-condition) has been proved by C. Hopkins [2, § 3].

Turning to the more general problem concerning the nilpotency of subrings, first note that the following useful lemma can be easily verified:

LEMMA. *If A is a nilpotent ideal of a ring S, and if each nil-subring of S/A is nilpotent, then also each nil-subring of S is nilpotent.*

Applying this lemma to a semi-primary ring (i. e., a ring S with a nilpotent radical N and with a semi-simple quotient-ring S/N), it follows directly from the author's theorem [4, § 2] that each nil-subring of a semi-primary ring is nilpotent (see [2, § 3]). On the other hand, applying our lemma to the final result of the present note (Theorem 6) we obtain the following theorem: *If A is a nilpotent ideal of a ring S, and if the M-condition holds for the r. ideals as well as for the l. ideals of S/A, then each nil-subring of S is nilpotent.* This last result is a considerable extension of

* Received January 18, 1945.

[1] G. Koethe [3, p. 165]. The numbers in brackets refer to the bibliography at the end of the paper.

[2] The abbreviations used in this paper are listed at the end of **1**.

[3] Some of the results of the present note were proved in the author's paper "On multiplicative systems" which was actually in the press for publication in the *Compositio Mathematica* in 1939; it does not seem to have appeared, presumably owing to war conditions.

the aforementioned result concerning semi-primary rings since, as is well known, the M-condition holds in a semi-simple ring for both $r.$ ideals and $l.$ ideals.[4]

Since an extensive use will be made here of the author's former results [5, § 2], it is convenient to adopt the terminology introduced in that paper. Thus a ring T is termed *semi-nilpotent*, if each subring of T which is generated by a finite set of elements, is nilpotent. A ring which is not semi-nilpotent is called *semi-regular*. The sum N of all semi-nilpotent ideals of a ring S is defined as the *radical* of the ring. The radical N is a semi-nilpotent ideal; it is a radical-ideal in the sense of Baer [1, § 1]; it contains all one-sided semi-nilpotent ideals of the ring, and the radical of S/N is zero. Finally, if $a \in S$, then $a \in N$ if and only if $aS \subseteq N$ (or $Sa \subseteq N$).

We shall use the notion of the right quotient $(A:K)_r$ which is defined, for an ideal A and an arbitrary subset K of S, as the set of all elements x of S satisfying the relation $K \cdot x \subseteq A$. This set is evidently a $r.$ ideal in S. Similarly the left quotient $(A:K)_l$ is defined as the set of all elements x satisfying the relation $x \cdot K \subseteq A$. An immediate consequence of the definition is the following property of the quotient: If $K \subseteq L$, then $(A:K)_r \supseteq (A:L)_r$, and $(A:K)_l \supseteq (A:L)_l$.

The following abbreviations are used in the present note: Maximal condition = M-condition; minimal condition = m-condition; two-sided ideal = ideal; right ideal = $r.$ ideal; left ideal = $l.$ ideal.

2. Semi-nilpotency of nil-rings. The theorems of this section are based on the properties of the radical [5, § 2] which were described in the previous section.

THEOREM 1. *Let S be an arbitrary ring, T a semi-regular nil-subring of S and N the radical of S. Then there exists an element a in T such that, on putting $R = a \cdot T$ and $L = T \cdot a$, we have:*

1) *The rings R, L, $R \cdot S$ and $S \cdot L$ are again semi-regular.*

2) $T \cdot S \supset R \cdot S$, $S \cdot T \supset S \cdot L$.

3) $(N:T)_l \subset (N:R)_l$, $(N:T)_r \subset (N:L)_r$.

4) *The $l.$ ideal $(N:R)_l$ and the $r.$ ideal $(N:L)_r$ are semi-regular.*

Proof. Denote by N^* the radical of T. Since T is semi-regular and N^* is semi-nilpotent, we have $N^* \subset T$. Now choose an element a such that $a \in T$,

[4] Further extensions of this result and others related to it will be discussed in a subsequent paper.

$a \notin N^*$; then it follows that a has the required properties. Indeed, on putting $R = a \cdot T$, $L = T \cdot a$, it follows (see [5, Theorem 5]) that R, as well as L, is semi-regular. Since the radical of T/N^* is zero, R^2 and L^2 must also be semi-regular, which in view of $R^2 \subseteq R \cdot S$, $L^2 \subseteq S \cdot L$ implies that $R \cdot S$ and $S \cdot L$ are also semi-regular. To prove 2) note that $0 \neq T \cdot S \supseteq R \cdot S$ and that from $T \cdot S = R \cdot S = a \cdot T \cdot S$ it would follow by left multiplication that $T \cdot S = a^k \cdot T \cdot S \neq 0$ for each k, which would imply that $a^k \neq 0$ for each k, in contradiction to the assumption that T was a nil-subring. Hence $T \cdot S \supset R \cdot S$, and similarly $S \cdot T \supset L \cdot T$. Turning to the proof of 3) first note that from $T \supset R$ it follows that

$$(N:T)_l \subseteq (N:R)_l. \tag{1}$$

Since, further, a is nilpotent, and $a \cdot T \not\subseteq N$ it follows that an integer k exists such that $k \supset 1$, $a^k T \subseteq N$ and $a^{k-1} \cdot T \not\subseteq N$, which implies that

$$a^{k-1} \notin (N:T)_l, \qquad a^{k-1} \in (N:R)_l. \tag{2}$$

From (1) and (2) follows the first part of 3). The second part of 3) can be proved similarly. As to the proof of 4), it follows from the definition of the quotient that $N \subseteq (N:T)_l$, and hence from the first inequality under 3) that we have $N \subset (N:R)_l$, which implies that $(N:R)_l$ is semi-regular. The semi-regularity of $(N:L)_r$ follows similarly.

THEOREM 2. *If in a ring S the M-condition or the m-condition holds for the semi-regular r. ideals then each nil-subring T of S is semi-nilpotent.*

Proof. Denote by N the radical of S, and suppose that T is a semi-regular ring, then by Theorem 1 there exist semi-regular subrings R_1 and L_1 in T such that $R_1 \cdot S$ and $(N:L_1)_r$ are semi-regular r. ideals in S satisfying the relations $T \cdot S \supset R_1 S$, $(N:T)_r \subset (N:L_1)_r$. Now by replacing in Theorem 1 the ring T by R_1 (or by L_1), we similarly deduce the existence of semi-regular subrings R_2 and L_2 such that $R_1 S \supset R_2 S$ and $(N:L_1)_r \subset (N:L_2)_r$. Proceeding in this manner we obtain, by induction, the infinite descending chain of semi-regular r. ideals

$$R_1 S \supseteq R_2 S \supseteq \cdots \tag{3}$$

and the infinite ascending chain of semi-regular r. ideals

$$(N:L_1)_r \cdot \subset (N:L_2)_r \subset \cdots. \tag{4}$$

By (3) and (4) it follows that as well the m-condition as the M-condition is incompatible with the supposed semi-regularity of T, i. e., that T is semi-nilpotent.

Since, in particular, either the M-condition or the m-condition of Theorem 2 implies that all nil-r. ideals and all nil-l. ideals are semi-nilpotent, it follows from Koethe's definition of the radical [3, § 3] and from Baer's definition of the upper radical [1, § 1] that

THEOREM 3. *If in a ring S the M-condition or the m-condition holds for the semi-regular r. ideals, then the radical in the sense of Koethe exists and is equal to the radical N of the ring. The radical N is then also identical with the upper radical in the sense of Baer.*

3. Chain-conditions for nilpotency of nil-subrings. In this section we shall prove *inter alia* the statements made in Section 1.

THEOREM 4. *If in a ring S the M-condition holds for the semi-nilpotent r. ideals, then the radical N of the ring S is nilpotent.*

Proof. From the M-condition it follows that N contains a finite set of elements $a_1, a_2, \cdots, a_m$ such that $N^2 = (a_1N, a_2N, \cdots, a_mN)$. Now denote by A the ring generated by the a_i; then we evidently have $N^2 = A \cdot N$. Multiplying this equation by N we obtain $N^3 = AN^2 = A^2N$. Proceeding in this manner with right multiplication by N we obtain, by induction,

$$N^k = A^{k-1}N, \qquad (k = 2, 3, \cdots). \tag{5}$$

Now since N is a semi-nilpotent ring, and since A is generated by a finite set of elements belonging to N, it follows that A is nilpotent, and by (5) that N also is nilpotent.

Remark. In a previous note it has been shown by the author [6, Theorem 4] that the nilpotency of N is also a consequence of the m-condition for the two-sided semi-nilpotent ideals of the ring. The foregoing theorem is an extension of the well-known fact that from the maximum condition for the r. ideals it follows that the sum of all nilpotent r. ideals is itself nilpotent.

The following theorem includes the solution of the problems of Baer and Koethe which were discussed in Section 1.

THEOREM 5. *Let N denote the radical of a ring S. If the M-condition holds for the r. ideals of S containing N (i. e., for the r. ideals of S/N) and for the r. ideals contained in N (i. e., for the semi-nilpotent r. ideals of S), then all nil-r. ideals and all nil-l. ideals of S are nilpotent.*

Proof. Denote by T any nil-r. ideal or nil-l. ideal of S; then $(T, N)/N$ is a nil-r. ideal or a nil-l. ideal in S/N. Now the radical of S/N is zero [5, Theorem 3], i. e., each non-zero one-sided ideal of S/N is semi-regular, and

since by Theorem 2 the nil-subring $(T, N)/N$ of S/N must be semi-nilpotent, it follows that $(T, N)/N$ must be the zero ideal of S/N, i. e., $T \subseteq N$. Now it follows, in view of Theorem 4, from the second condition of our theorem, that N is nilpotent, and since T is a subset of N, that T also is nilpotent.

Remark. An analogous theorem for the m-condition can be derived as a consequence of the author's former result [6, Theorem 5]. However, we can now improve this result as follows:

THEOREM 5a. *Let N denote the radical of a ring S. If the m-condition holds for the nil-r. ideals containing N, and for the ideals contained in N, then all nil-r. ideals and all nil-l. ideals of S are nilpotent.*

Proof. Indeed it is easily verified that if the semi-regular subring T of Theorem 1 is a nil-r. ideal or a nil-l. ideal, then the rings $R \cdot S$ and $S \cdot L$ of that theorem are a nil-r. ideal and a nil-l. ideal, respectively. Accordingly, we may assume that in this case the r. ideals of the descending chain (3) in the proof of Theorem 2, are all nil-r. ideals. These remarks, combined with the author's former result [6, Theorem 4] yield, as in the proof of Theorem 5, the required result.

Turning now to the problem of nilpotency of subrings, we prove

THEOREM 6. *If the M-condition holds for the r. ideals as well as for the l. ideals of a ring S, then each nil-subring T of S is nilpotent.*

Proof. It follows from the M-condition for r. ideals that there exists in T a finite set of elements $a_1^{(1)}, a_2^{(1)}, \cdots, a_{n_1}^{(1)}$ such that $T \cdot S = (a_1^{(1)}S, a_2^{(1)}S, \cdots, a_{n_1}^{(1)}S)$. Denote by A_1 the ring generated by the $a^{(1)}$; then we evidently have $T \cdot S = A_1 S$. Similarly it follows that T contains a finite set of elements $a_1^{(2)}, a_2^{(2)}, \cdots, a_{n_2}^{(2)}$ such that $T^2S = (a_1^{(2)}TS, \cdots, a_{n_2}^{(2)}TS)$. Denoting by A_2 the ring generated by the $a^{(2)}$, we obtain $T^2S = A_2T \cdot S = A_2 \cdot A_1 \cdot S$. Proceeding in this manner we obtain, by induction, an infinite sequence of rings $A_1, A_2, \cdots$, each of which is generated by a finite set of elements of T, such that we have

$$T^kS = A_kA_{k-1} \cdots A_1S, \qquad (k = 1, 2, \cdots). \tag{6}$$

Now suppose that

$$T^k \neq 0, \qquad (k = 1, 2, \cdots); \tag{7}$$

then by (6) in view of $T^kS \supseteq T^{k+1}$ we would have

$$A_kA_{k-1} \cdots A_1 \neq 0, \qquad (k = 1, 2, \cdots). \tag{8}$$

Since, by Theorem 2, the ring T is semi-nilpotent, it follows by the definition

of the A that each A_k is nilpotent. Hence by (8) we deduce the existence of a positive integer $s_k = s$ such that

$$A_k^s A_{k-1} \cdot \cdot \cdot A_1 S = 0, \qquad A_k^{s-1} A_{k-1} \cdot \cdot \cdot A_1 S \neq 0, \qquad s > 1,$$

which implies that

$$(9) \qquad A_k^{s-1} \subsetneqq (0 : A_{k-1} \cdot \cdot \cdot A_1 S)_l, \qquad A_k^{s-1} \subseteq (0 : A_k A_{k-1} \cdot \cdot \cdot A_1 S)_l,$$

i. e., $(0 : A_{k-1} \cdot \cdot \cdot A_1 S) \neq (0 : A_k A_{k-1} \cdot \cdot \cdot A_1 S)$. On the other hand, it follows from (6) in view of $T^k \cdot S \subseteq T^{k-1} \cdot S$ that

$$(10) \qquad (0 : A_{k-1} \cdot \cdot \cdot A_1 S)_l \subseteq (0 : A_k \cdot \cdot \cdot A_1 S)_l.$$

By (9) and (10 we finally obtain

$$(11) \qquad (0 : A_{k-1} \cdot \cdot \cdot A_1 \cdot S)_l \subset (0 : A_k A_{k-1} \cdot \cdot \cdot A_1 \cdot S)_l, \qquad (k = 2, 3, \cdot \cdot \cdot),$$

i. e., an infinite ascending chain of l. ideals, which is a contradiction to the M-condition for l. ideals. Hence, the supposition (7) is false, i. e., T is nilpotent.

Remark. Applying the lemma of **1** we can extend Theorem 6 if we replace its conditions by assumptions analogous to those of Theorem 5.

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UNIVERSAL FUNCTIONS OF POLYGONAL NUMBERS, III.*

By L. W. Griffiths.

1. Introduction. The universal functions of polygonal numbers of order $m+2$, with the sum of the coefficients in each function equal to $m+4$, are determined in this paper. If the sum of the coefficients is at most $m+3$ the universal functions have been determined.[1] A necessary and sufficient condition that the universality of a function of weight $m+4$ be not implied by that of a function of weight at most $m+3$ is proved. For each m such that $m \geqq 3$ a universal function of weight $m+4$ satisfying this condition is exhibited.

In this paper use is made of facts on simultaneous representation due to Dickson,[2] and new results are proved for the case $(1, 2, 2, 3)$ with b divisible by four.

The general development of this paper is suggested by the first paper to which reference has been made. Certain notations and facts established in that paper are used. Many proofs are so similar that they are not presented here.

The polygonal numbers of order $m+2$ are defined, for m a positive integer, by $p(x) = x + m(x^2 - x)/2$ with $x = 0, 1, 2, \cdots$. Here $m \geqq 3$, for the reasons stated in I. The coefficients $a_1, \cdots, a_n$ in the functions $f = (a_1, \cdots, a_n) = a_1 p_1 + \cdots + a_n p_n$ are positive integers to be determined. Also $1 \leqq a_1 \leqq \cdots \leqq a_n$, and, by definition, $w_k = a_1 + \cdots + a_k$ $(1 \leqq k \leqq n)$ and $w = w_n$. It will be proved that, if $w = m + 4 \geqq 7$, then f is universal only if f satisfies (1), or (2), or (3), or (4), or (5) and (7), or (6) and (7). It will also be proved that f is universal if f satisfies (1) or (2) or (4). If f satisfies (3), or (5) and (7), or (6) and (7), then f represents every positive integer A except perhaps when A is in the specific range stated in Theorem 6. This range depends only on m and f. The methods of this paper are applicable to the verification that f actually represents these integers A. Verification is practically certain, and is being carried forward as assistance is available.

* Received November 30, 1944.

[1] L. W. Griffiths, *Annals of Mathematics*, vol. 31 (1930), pp. 1-12; *American Journal of Mathematics*, vol. 66 (1944), pp. 97-100. These papers will be cited as I and II respectively.

[2] L. E. Dickson, *American Journal of Mathematics*, vol. 56 (1934), pp. 513-528.

2. Necessary and sufficient conditions that f represent integers less than $145m+25$. The following lemmas and theorems have been proved by methods similar to those used in the proofs of the corresponding facts in I.

LEMMA 1. *If $w = m+4$, then $f = 0, \cdots, m+1$ if and only if $f = (1, a_2, \cdots, a_n)$ and $a_k \leqq w_{k-1}+1$ $(2 \leqq k \leqq n)$.*

LEMMA 2. *If $w = m+4$ and if $f = (1, 1, \cdots, a_n)$, then $f = m+5, \cdots, 2m+3$ if and only if $a_k \leqq w_{k-1}$ $(3 \leqq k \leqq n)$.*

LEMMA 3. *If $w = m+4 \geqq 8$ and if $f = (1, 1, \cdots, a_n)$, then $f = 2m+6, \cdots, 3m+2$ if and only if $a_3 = 1$ or 2 and $a_k \leqq w_{k-1}-1$ $(4 \leqq k \leqq n)$.*

Theorem 1 follows from Lemmas 1, 2, and 3, and direct verification for the integers $m+2$, $m+3$, $m+4$, $2m+4$, $2m+5$, $3m+3$, if $m \geqq 4$; if $m = 3$ it follows from Lemmas 1 and 2, and direct verification for the integers $2(m+2), \cdots, 3m+3$.

THEOREM 1. *If $w = m+4 \geqq 7$ and if $f = (1, 1, \cdots, a_n)$, then $f = 0, \cdots, 3m+3$ if and only if $a_3 = 1$ or 2 and $a_k \leqq w_{k-1}-1$ $(4 \leqq k \leqq n)$.*

If f satisfies the conditions of Theorem 1 with $a_3 = 1$, then it represents the integers $3m+4, \cdots, 6m+8$ if and only if it is not one of the functions $(1, 1, 1, 1, 3, \cdots, a_n)$ with $n \geqq 6$. Each of these functions does not represent $5m+9$. If f satisfies the conditions of Theorem 1 with $a_3 = 2$, then $f = 3m+4, \cdots, 5m+7$ if and only if f is the function $(1, 1, 2, 3)$ or one of the functions $(1, 1, 2, a_4, \cdots, a_n)$ with $a_4 = 2$ or 3 and $a_k \leqq w_{k-1}-2$ $(5 \leqq k \leqq n)$. If f satisfies the conditions of Theorem 1 with $a_3 = 2$ and if there is an integer k such that $k \geqq 5$ and $a_k = w_{k-1}-1$, then f does not represent $4m+6+a_k+\cdots+a_n$. Each of the functions $(1, 1, 2, 2, a_5, \cdots, a_n)$ with $a_k \leqq w_{k-1}-2$ $(5 \leqq k \leqq n)$ represents $5m+8, \cdots, 6m+8$. Each of the functions $(1, 1, 2, 3, a_5, \cdots, a_n)$ with $a_k \leqq w_{k-1}-2$ $(5 \leqq k \leqq n)$ does not represent $5m+9$. The function $(1, 1, 2, 3)$ has $w = 7$, and hence $m = 3$ and $5m+7 = 6m+4$; it represents $6m+5, \cdots, 6m+8$ by direct verification.

THEOREM 2. *If $w = m+4 \geqq 7$ and if $f = (1, 1, \cdots, a_n)$, then $f = 0, \cdots, 6m+8$ if and only if f satisfies* (1), (2), (3), *or* (4):

(1) $f = (1, 1, 1, 1, a_5, \cdots, a_n)$, $a_k \leqq w_{k-1}-1$ $(5 \leqq k \leqq n)$, *and* $a_5 \neq 3$ *if* $n > 5$,

(2) $f = (1, 1, 1, 2, a_5, \cdots, a_n)$, $a_k \leqq w_{k-1}-1$ $(5 \leqq k \leqq n)$,

(3) $f=(1,1,2,2,a_5,\cdots,a_n)$, $\quad a_k \leqq w_{k-1}-2$ $(5 \leqq k \leqq n)$,

(4) $f=(1,1,2,3)$.

LEMMA 4. *If $w=m+4$ and if $f=(1,2,a_3,\cdots,a_n)$, then $f=3m+4$ if, and only if, $a_3=2$.*

LEMMA 5. *If $w=m+4$ and if $f=(1,2,a_3,\cdots,a_n)$, then $f=m+5,\cdots,2m+3$ if and only if there is an integer J such that $2 \leqq J \leqq n-1$ and $a_{J+1}=3$, and that if $J<n-1$ then $a_k \leqq w_{k-1}-2$ $(J+2 \leqq k \leqq n)$, and that if there is an integer k such that $k>J+1$ and $a_k=w_{k-1}-3$ then there is an integer K such that $a_{K+1}=a_k+1$.*

If f satisfies the conditions of Lemmas 4 and 5 then, by the hypothesis that the coefficients are arranged so that $a_1 \leqq a_2 \leqq \cdots \leqq a_n$, it is true that f satisfies the conditions of Lemma 1. By direct verification, such an f represents $m+2$, $m+3$, $m+4$, $2m+4$, $2m+5$, $3m+3$.

LEMMA 6. *If $w=m+4$ and if $f=(1,2,a_3,\cdots,a_n)$, then $f=2m+6,\cdots,3m+2$ if and only if $a_k \leqq w_{k-1}-1$ $(3 \leqq k \leqq n)$.*

Theorem 3 follows from Lemmas 1, 4, 5, and 6.

THEOREM 3. *If $w=m+4 \geqq 7$ and if $f=(1,2,a_3,\cdots,a_n)$, then $f=0,\cdots,3m+4$ if, and only if, $a_3=2$ and there is an integer J such that $3 \leqq J \leqq n-1$ and $2=a_3=\cdots=a_J$ and $a_{J+1}=3$ and that the remaining coefficients satisfy the conditions of Lemma 5.*

If f satisfies the conditions of Theorem 3 and if there is an integer k such that $k \geqq J+2$ and $a_k=w_{k-1}-2$, then f does not represent $4m+7+a_k$. If f satisfies the conditions of Theorem 3 with $a_k \leqq w_{k-1}-3$ $(J+2 \leqq k \leqq n)$, then $f=3m+5,\cdots,7m+10$. If the function $(1,2,2,3,a_5,\cdots,a_n)$ satisfies the conditions of Theorem 3 and has $a_k \leqq w_{k-1}-3$ $(J+2 \leqq k \leqq n)$, then it represents $7m+12$ if, and only if, there is an integer S such that $S \geqq 5$ and $a_S=4$. The function $(1,2,2,3)$ represents $7m+12$ but does not represent $8m+12$. If the function $(1,2,2,2,a_5,\cdots,a_n)$ satisfies the conditions of Theorems 3 and has $a_k \leqq w_{k-1}-3$ $(J+2 \leqq k \leqq n)$, then it represents the integers $7m+11,\cdots,17m+14$. If the function $(1,2,2,3,a_5,\cdots,a_n)$ satisfies the conditions of Theorem 3 and if there is an integer S such that $S>4$ and $a_S=4$, and either $S=n$ or $a_k \leqq w_{k-1}-3$ $(S+1 \leqq k \leqq n)$, then it represents the integers $7m+11,\cdots,17m+14$. If the function $(1,2,2,2,3,7,a_7,\cdots,a_n)$ with $n \geqq 7$ satisfies the conditions of Theorem 3 and has $a_k \leqq w_{k-1}-3$ $(J+2 \leqq k \leqq n)$, it does not represent $17m+16$; the same fact is true of the function $(1,2,2,2,3,5,a_7,\cdots,a_n)$ if $a_k \neq 6$

$(7 \leqq k \leqq n)$. It has been proved by direct verification that each of the remaining functions, that is, each function which satisfies the conditions of Theorem 3 and which represents $3m+5, \cdots, 17m+16$, does indeed represent $17m+17, \cdots, 147m+37$. These facts and Theorem 3 prove Theorem 4.

THEOREM 4. *If* $w = m+4 \geqq 7$ *and if* $f = (1, 2, a_3, \cdots, a_n)$, *then* $f = 0, \cdots, 147m+37$ *if, and only if,* f *satisfies* (5) *and* (7), *or* (6) *and* (7):

(5) $f = (1, 2, 2, 2, a_5, \cdots, a_n)$, $2 = a_4 = \cdots = a_J$, $a_{J+1} = 3$, $J = n-1$ *or* $a_k \leqq w_{k-1} - 3$ $(J+2 \leqq k \leqq n)$, *but* f *is not* $(1, 2, 2, 2, 3, 7, \cdots, a_n)$ *with* $n \geqq 7$ *and* f *is not* $(1, 2, 2, 3, 5, \cdots, a_n)$ *with* $n \geqq 7$ *and* $a_k \neq 6$ $(7 \leqq k \leqq n)$.

(6) $f = (1, 2, 2, 3, a_5, \cdots, a_n)$, $3 = a_4 = \cdots = a_{S-1}$, $a_S = 4$, $S = n$ *or* $a_k \leqq w_{k-1} - 3$ $(S+1 \leqq k \leqq n)$,

(7) *if there is a coefficient* a_k *such that* $a_k = w_{k-1} - 3$ *then there is a coefficient greater than* a_k *and the first such coefficient is indeed* $a_k + 1$.

It has been proved by direct verification that: if f satisfies (1) then $f = 6m+9, \cdots, 202m+28$; if f satisfies (2) then $f = 6m+9, \cdots, 387m+116$; if f satisfies (3) or (4) then $f = 6m+9, \cdots, 145m+25$.

3. Universality of the functions in Theorem 2 and of the functions in Theorem 4. Theorem 5 has been established by methods similar to those used in the proofs of the corresponding facts in I. Use was made of facts on simultaneous representation due to Dickson,[2] of results equivalent to his for the case $(1, 2, 2, 2)$ stated in Lemma 9, and of new results for the case $(1, 2, 2, 3)$ with b divisible by four stated in Lemma 8.

THEOREM 5. *If* f *satisfies the conditions of Theorem* 2 *or the conditions of Theorem* 4, *then there is a positive integer* M, *depending only on* m *and* f, *such that* f *represents every integer which is greater than* M. *If* f *satisfies* (1) *with* $a_5 = 1$, *then* $M = 44m+40$; *if* f *satisfies* (1) *with* $a_5 = 2$, *or if* f *is* $(1, 1, 1, 1, 3)$, *then* $M = 142m+76$; *if* f *satisfies* (2) *then* $M = 387m+116$; *if* f *is* $(1, 1, 2, 3)$ *then* $M = 73m+90$. *If* f *satisfies* (3) *and the value stated in Lemma* 7 *is used for* d, *then* $M = [11d(d-5)+74]m + 2(11d-29)$; *if* f *satisfies* (5) *and* (7), *then* $M = [13d(d-5)+87]m + 2(13d-379)$; *if* f *satisfies* (6) *and* (7), *then* $M = [15d(d-5)+100]m + 2(15d-55)$.

LEMMA 7. *If* f *satisfies* (3) *then for each function the corresponding value of* d *appears in the following list:* $d = 2a_5 + 4$ *if* $a_5 = 3$ *or* 4; $d = 8$

if $a_5 = 2$ *and* $n = 5$; $d = 12$ *if* $a_5 = 2$ *and* $a_6 = 2$; $d = 2a_6 + 4$ *if* $a_5 = 2$ *and* $a_6 \geqq a_5 + 1$. *If* f *satisfies* (5) *with* $a_5 = 2$, *or with* $a_5 = 3$ *and* $a_6 = 5$, *then* $d = 9$; *if* f *satisfies* (5) *with* $a_5 = 3$, *and if* $n = 5$ *or* $a_6 = 3$ *or* $a_6 = 6$, *then* $d = 10$; *if* f *satisfies* (5) *with* $a_5 = 3$, *and if* $a_6 = 4$ *or* $a_6 = 7$, *then* $d = 12$. *If* f *satisfies* (6), *then for each function the corresponding value of* d *appears in the following list:* $d = 10$ *if* $a_5 = 3$; $d = 24$, $a_5 = 4$ *and* $n = 5$; $d = 2a_6 + 4$ *if* $a_5 = 4$ *and* $a_6 = 5$, 7, 9, *or* 6; $d = 28$ *if* $a_5 = 4$ *and* $a_6 = 8$, *or if* $a_5 = 4 = a_6$ *and* $n = 6$, *or if* $a_5 = 4 = a_6$ *and* $a_7 = 4$, 8, *or* 12; $d = 20$ *if* $a_5 = 4 = a_6$ *and* $a_7 = 5$, 6, *or* 7; $d = 2a_7 + 4$ *if* $a_5 = 4 = a_6$ *and* $a_7 = 9$, 10, 11, or 13.

Theorem 6 follows from Theorems 2, 4, and 5, and the facts stated at the end of Section 2.

THEOREM 6. *If* f *satisfies* (1), *or* (2), *or* (4), *then* f *represents every positive integer. If* f *satisfies* (3) *then* f *represents every positive integer* A *except perhaps when* $145m + 25 < A < M$. *If* f *satisfies* (5) *and* (7), *or if* f *satisfies* (6) *and* (7), *then* f *represents every positive integer except perhaps when* $147m + 37 < A < M$. *The values of* M *are given in Theorem* 5.

The universality of a function of weight $m + 4$ is implied by that of a function of weight $m + 3$ if and only if the coefficients of the former are obtained from the coefficients of the latter by prefixing a coefficient 1. The universality of a function of weight $m + 4$ is implied by that of a function of weight $m + 2$ if and only if the coefficients of the former are obtained from those of the latter by inserting a coefficient 2 or by prefixing two coefficients each equal to 1. It is thus proved, by comparison of the functions of Theorems 2 and 4 with the universal functions listed in I and II, that the universality of a function of weight $m + 4$ is implied by that of a function of weight $m + 2$ or that of a function of weight $m + 3$ if and only if f satisfies one of the conditions in the following list: f satisfies (2) with $a_5 = 2$, and with $n = 5$ or $a_k \leqq w_{k-1} - 3$ $(6 \leqq k \leqq n)$; f satisfies (1) with $a_5 = 2$ and $a_k \leqq w_{k-1} - 2$ $(6 \leqq k \leqq n)$; f satisfies (1) with $a_5 = 1$, and either $a_6 = 3$ and $n = 6$, or $a_6 = 2$ and $n = 6$ or $a_k \leqq w_{k-1} - 2$ $(7 \leqq k \leqq n)$, or $a_6 = 1$ and $a_k \leqq w_{k-1} - 2$ $(7 \leqq k \leqq n)$. These facts prove Theorem 7.

THEOREM 7. *The universality of a function of weight* $m + 4$ *is not implied by that of a function of weight* $m + 2$ *and is not implied by that of a function of weight* $m + 3$ *if and only if the function satisfies* (3), *or* (4), *or* (5) *and* (7), *or* (6) *and* (7), *or it satisfies* (1) *or* (2) *and is not in the preceding list.*

If $m \geqq 9$ the function f defined as follows satisfies Theorem 7. Define q

and r by $m-1=4q+r$, with $0 \leqq r \leqq 3$, and let $a_1=1=a_2=\cdots=a_5$, $a_6=4=\cdots=a_{q+4}$, $a_{q+5}=r+4$. Then f satisfies (1) with $a_5=1$ and $a_6=4$, and hence f is non-trivially universal. If $m=3,\cdots,8$ the following functions, respectively, are universal and satisfy Theorem 7: $(1,1,2,3)$, $(1,1,1,2,3)$, $(1,1,1,1,1,4)$, $(1,1,2,2,4)$, $(1,1,1,1,1,3,3)$, $(1,1,1,1,1,3,4)$.

4. Lemmas on simultaneous representation for the case (1, 2, 2, 3) with b divisible by four and for the case (1, 2, 2, 2). Dickson[2] proved that if a and b are integers such that

$$(8) \qquad a \equiv b \pmod 2, \qquad b^2 \leqq 8a, \qquad 7a \leqq b^2+2b+7, \qquad b \geqq 0,$$

and if b is not a multiple of four, then there are integers x, y, z, and w, such that each is non-negative and that

$$(9) \qquad a=x^2+2y^2+2z^2+3w^2, \qquad b=x+2y+2z+3w.$$

The case in which b is a multiple of four is treated in Lemma 8.

LEMMA 8. *If a and b are integers such that* (8) *holds, and if i and h are integers such that $i \geqq 2$ and $h \geqq 1$, and if*

$$(10) \qquad b=2^iB, \qquad a=a^{2^{h-1}}A \text{ or } 2^{2h}A, \qquad A \text{ and } B \text{ odd},$$

then (9) *have a solution in integers, each non-negative, if and only if no one of the conditions* (11), (12), (13), (14), (15), *is satisfied:*

$$(11) \qquad a=2^{2h}A \text{ and } i=h+1, \text{ and } A \equiv 3 \pmod 4,$$

$$(12) \qquad a=2^{2h-1}A \text{ and } i=h+1 \text{ and } A-B^2 \text{ of the form } 4^t(8s+5),$$

$$(13) \qquad a=2^{2h-1}A \text{ and } i \geqq h+3 \text{ and } A \equiv 5 \pmod 8,$$

$$(14) \qquad a=2^{2h-1}A \text{ and } i=h+2 \text{ and } A \equiv 1 \pmod 8,$$

$$(15) \qquad a=2^{2h-1}A \text{ and } i>h=1 \text{ and } A \equiv 1 \pmod 4.$$

The proof of this lemma is based on the fact, which Dickson proved in Section 4 of the paper to which reference has been made, that there are integers x, y, z, w satisfying (9) if and only if there are integers V, D, W such that

$$(16) \qquad 8a-b^2=V^2+8D^2+6W^2,$$

$$(17) \qquad b+V-4D \equiv 0 \pmod 8, \qquad b-2D-W \equiv 0 \pmod 4.$$

The proof also uses the fact that if N is an odd integer then there are integers u and v which are both odd such that $4N=u^2+3v^2$ if, and only if, there

are integers u_1 and v_1 which are both even such that $4N = u_1^2 + 3v_1^2$. Hence, by using (10) in (16) and an exhaustive separation into cases, it is proved that, if there are integers V, D, W satisfying (16), then there are integers x, y, z, w satisfying (9) if and only if (15) does not hold. Now $8a - b^2$ is a multiple of four, and a multiple of four is represented by the form $V^2 + 8D^2 + 6W^2$ if and only if[3] it is not of the form $4^t(8s + 5)$. By using (10) it is proved that $8a - b^2 = 4^t(8s + 5)$ if and only if one of the conditions (11)-(14) is satisfied. All these facts prove Lemma 8.

Dickson proved that if a and b are integers such that

$$(18) \qquad a \equiv b \pmod 2, \quad b^2 \leqq 7a, \quad 6a \leqq b^2 + 2b + 6, \quad b \geqq 0,$$

and if $7a - b^2$ is not of the form $4^t(16s + 14)$, then there are integers x, y, z, w, such that each is non-negative and that

$$(19) \qquad a = x^2 + 2y^2 + 2z^2 + 2w^2, \quad b = x + 2y + 2z + 2w.$$

If the notations (10) are used in the equation $7a - b^2 = 4^t(16s + 14)$, the conditions stated in Lemma 9 are verified.

LEMMA 9. *If a and b are integers such that* (18) *holds, and if i and h are integers such that $i \geqq 0$ and $h \geqq 0$ and the notations* (10) *are used, then* (19) *have a solution in integers, each non-negative, if and only if no one of conditions* (20), (21), (22), (23) *is satisfied:*

(20) $a = 2^{2h}A$ *and* $i = h \geqq 0$, *and* $A \equiv 1 \pmod{16}$ *and* $B \equiv \pm 3 \pmod 8$,

(21) $a = 2^{2h}A$ *and* $i = h \geqq 0$, *and* $A \equiv 9 \pmod{16}$ *and* $B \equiv \pm 1 \pmod 8$,

(22) $a = 2^{2h-1}A$ *and* $i = h \geqq 1$, *and* $A \equiv 3 \pmod 8$,

(23) $a = 2^{2h-1}A$ *and* $i \geqq h + 1 \geqq 2$, *and* $A \equiv 1 \pmod 8$.

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[3] B. W. Jones, *University of Chicago Dissertation* (1928), p. 122.

THE HOMOMORPHISM THEOREMS FOR LOOPS.*

By REINHOLD BAER.

A loop is a system L of elements with one composition, called addition, $x+y$, subject to the following rules:

I. If two of the three elements x, y, z are given, then the equation $x+y=z$ has one and only one solution in L.

II. There exists a nul-element 0, satisfying $x+0=0+x=x$.

Thus addition, right- and left-subtraction are unique; and this makes the term subloop self-explanatory.

A homomorphism is a single-valued and addition-preserving mapping of one loop into (the same or another) loop. It has been noted that in the theory of homomorphisms classical group-theory does not make much use of the associative law; and the present note is intended to contribute towards a substantiation of this contention.

The set of all elements in a loop which are mapped upon 0 by a given homomorphism is a subloop which is termed the kernel of this homomorphism; and every subloop which is the kernel of some homomorphism [1] has been called a normal subloop. The following theorems are due to A. A. Albert and M. F. Smiley.[2]

The subloop S of the loop L is a normal subloop if, and only if,

$$(1)\qquad \begin{aligned} x+S=S+x, \qquad (x+y)+S=x+(y+S),\\ (x+S)+y=x+(S+y), \qquad (S+x)+y=S+(x+y) \end{aligned}$$

for x and y in L.

If S is a normal subloop of the loop L, then the sets $S+x$ for x in L are termed the cosets of L modulo S. These cosets form the quotient loop L/S; and mapping the element x in L upon the coset $S+x$ constitutes the natural homomorphism of L upon L/S.

The isomorphism theorems for quotient groups including Zassenhaus' four subgroups formula have been proved for loops too.[3] But it seems to the

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[1] Most writers on loop theory define normality by this or an equivalent property.

[2] Cf. Smiley (1), Theorem 2, p. 783.

[3] Albert (2), Theorems 4 and 5, pp. 404-405; Smiley (1), Theorem 5, p. 784.

author that Zassenhaus' proof[4] is still the simplest and most advantageous one, though it has, apparently, not yet been used for loops. This proof is offered here for two reasons: firstly some slight changes are necessitated through omission of the associative law; and thus it may be convenient to have available a detailed exposition. Secondly it forms the basis for a deeper investigation of these laws; and we are able to prove both extensions and converses of them which are new in the associative case too.

The distinctive feature of the method used here is a preference for the use of homomorphisms instead of computations with elements. Thus of the two definitions of normal subloops mentioned above we neglect the one which expresses itself in the equations (1) and we utilize mainly the characterization as the kernel of a homomorphism.

The isomorphism laws are known to lead immediately to a proof of the Jordan-Hölder-Schreier-Zassenhaus Theorem. We give an indication of this proof too, since the frame work given is slightly more general than the one customarily used, and since in the case of loops there occur some possibilities beyond those encountered in the classical case.

Notations. If S and T are subsets of the loop L, then we denote by $S+T$ the *set* of all the elements $s+t$ for s in S and t in T; by $\{S,T\}$ the subloop generated by S and T, i. e., the smallest subloop of L containing S and T; and by $S \cap T$ the cross cut of S and T.

If x is an element in the loop L, and if η is a homomorphism of L, the image of x under η is denoted by $x\eta$; and likewise we denote by $S\eta$ the image of the subset S of L. We say that η is a homomorphism of L *into* H, if $L\eta$ is a subset, and therefore a subloop, of the loop H; and if $L\eta = H$, then η is termed a homomorphism of L *upon* H. As usual an isomorphism is a homomorphism which maps different elements upon different elements.

1. The first isomorphism theorem.

THEOREM 1. *If S and T are subloops of the loop L, and if S is a normal subloop of $\{S,T\}$, then*

(a) $\{S,T\} = S+T = T+S$;

(b) *$S \cap T$ is a normal subloop of T;*

(c) *reciprocal isomorphisms between $\{S,T\}/S$ and $T/(S \cap T)$ are obtained by mapping the coset X of $\{S,T\}/S$ upon the set $X \cap T$ and by mapping the coset Y of $T/(S \cap T)$ upon the set $S+Y$.*

[4] Zassenhaus (1), pp. 32-33 and pp. 50-52. Of course, Zassenhaus is concerned only with the associative case.

Proof. (a) may either be verified by reference to the characteristic equations (1), stated in the introduction, or by the following argument. Since S is a normal subloop of the loop $K = \{S, T\}$, there exists a homomorphism η of K (upon some loop J) whose kernel is exactly S. Clearly $K\eta = \{S\eta, T\eta\} = T\eta$. Hence there exists to every element x in K an element x^* in T such that $x\eta = x^*\eta$. Consequently x and x^* belong [5] to the same coset of K modulo S so that x belongs to $S + x^* = x^* + S$. Since x^* is an element in T, we have found that $K = S + T = T + S$.

The homomorphism η of K defines a homomorphism of the subloop T of K (upon the subloop $T\eta$ of H). The kernel of this homomorphism of T is obviously the cross cut $S \cap T$ of T and of the kernel S of η. Thus $S \cap T$ is a normal subloop of T.

We have noted before that $K\eta = T\eta = J$ so that $\{S, T\}$ and T have the same image under the homomorphism η. If y is any element in J, then denote by y' the set of all the elements in K that are mapped by η upon y; and denote by y'' the set of all the elements in T that are mapped by η upon y. Clearly $y'' = T \cap y'$ and $y' = S + y''$; and (c) is an immediate consequence of these facts.

LEMMA. *If N is a normal subloop of the loop L, if η is a homomorphism of L, and if $N\eta = 0$, then η defines a homomorphism of L/N.*

Proof. If x and y are any two elements in L, then there exists one and only one element z in L such that $z + y = x$. If x and y belong to the same coset modulo N, then z is in N (and conversely). In this case $y\eta = z\eta + y\eta = (z + y)\eta = x\eta$; and our contention is an immediate consequence of this fact.

THEOREM 2. *If M and N are normal subloops of the loops K and L, respectively, and if η is a homomorphism of K into L such that $M\eta \leq N$, then a homomorphism of K/M into L/N is defined by mapping the coset X of K/M upon $N + X\eta$.*

Proof. Denote by ν the natural homomorphism of L upon L/N which maps the element y in L upon the coset $N + y$. Since the product of two homomorphisms is a homomorphism, if defined, $\eta\nu$ is a homomorphism of K into L/N. From $M\eta \leqq N$ we infer $M\eta\nu = 0$; and hence it follows from Lemma 1 that $\eta\nu$ defines a homomorphism of K/M into L/N. Clearly $X\eta\nu = N + X\eta$ for X in K/M, completing the proof.

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[5] The uniquely determined solutions y, z of the equations $y + x^* = x$ and $x^* + z = x$ belong to S.

Theorem 2 leads us to the following convenient *definition*: If M and N are normal subloops of the loops K and L respectively, and if η is a homomorphism of K into L such that $M\eta \leq N$, then *the homomorphism of K/M into L/N which is induced by η* maps X in K/M upon $N + X\eta$. It is clear that η cannot induce a homomorphism of K/M into L/N, unless $M\eta \leq N$; and thus the statement: "η induces a homomorphism of K/M into L/N" shall always be understood to imply $M\eta \leq N$.

It may be worth mentioning that the three Isomorphism Theorems are either contained in or may be inferred from the results obtained so far.

THEOREM 3. *The following properties of the subloops R and S of the loop L and of the normal subloops M and N of R and S, respectively, imply each other.*

(i) $N = M \cap S$ *and* $R = M + S$.

(ii) *The identity automorphism of L induces an isomorphism of S/N upon R/M.*

(iii) $N \leq M$, $S \leq R$; *and there exists a homomorphism η of L, a subloop T of $L\eta$ and a normal subloop P of T such that η induces both an isomorphism of R/M upon T/P and an isomorphism of S/N upon T/P.*

Proof. It is an immediate consequence of Theorem 1 (c) that (i) implies (ii). If (ii) holds true, then let $\eta = 1$, $R = T$, $M = P$; and it is evident that (ii) implies (iii).

Assume finally the validity of (iii). Denote by A the set of all the elements in L which are mapped by η upon elements in T; and denote by B the set of all the elements in L which are mapped by η upon elements in P. Clearly A and B are subloops of L. Mapping x in A upon the coset $P + x\eta$ in T/P we obtain a homomorphism of A into T/P whose kernel is B, showing that B is a normal subloop of A. Since η is supposed to induce an isomorphism of R/M upon T/P, we have $N \leq M \leq B$ and $S \leq R \leq A$. The only elements in R which are mapped by η upon elements P are the elements in M, implying $M = B \cap R$; and likewise one deduces $N = B \cap S$ from the fact that η induces an isomorphism of S/N upon T/P. Consequently we have:

$$N \leq M \cap S \leq B \cap S = N \quad \text{or} \quad N = M \cap S.$$

If r is an element in R, then $P + r\eta$ is a coset of T/P, since η induces an isomorphism of R/M upon T/P. But η induces an isomorphism of S/N upon T/P; and hence there exists (Theorem 2) an element s in S such that $P + r\eta = P + s\eta$. Consequently there exists an element p in P such that

$r\eta = p + s\eta$. There exists furthermore one and only one element q in R such that $r = q + s$, since r and s are both in the subloop R of L. Hence

$$p + s\eta = r\eta = (q + s)\eta = q\eta + s\eta, \quad \text{proving} \quad p = q\eta.$$

Hence q belongs to the cross cut $B \cap R = M$; and $r = q + s$ belongs to $M + S$. Thus we have shown that $M + S \leq R \leq M + S$ or $R = M + S$, proving that (i) is a consequence of (iii).

COROLLARY 1. *If M and S are normal subloops of the subloop R of the loop L, if N is a normal subloop of S, and if conditions (i) to (iii) of Theorem 3 are satisfied by M, N, R, S, then N is a normal subloop of M and M/N and R/S are isomorphic loops.*

This is an immediate consequence of Theorem 1 and of the symmetry of condition (i).

THEOREM 4. *The following properties of the subloops R and S of the loop L and of the normal subloops M and N of R and S, respectively, imply each other.*

(i) *$N \leq M$ and $R = M + S$.*

(ii) *The identity automorphism of L induces a homomorphism of S/N upon R/M.*

(iii) *$N \leq M$, $S \leq R$; and there exists a homomorphism η of L, a subloop T of $L\eta$ and a normal subloop P of T such that η induces an isomorphism of R/M upon T/P and a homomorphism of S/N upon T/P.*

Proof. If (i) holds true, then mapping the coset X of S/N upon $M + X$ constitutes the homomorphism of S/N upon R/M which is induced by the identity automorphism of L, proving that (ii) is a consequence of (i).

To prove that (iii) is a consequence of (ii), one puts $\eta = 1$, $R = T$ and $M = P$.

If finally (iii) is satisfied, then denote by E the set of all the elements in S which are mapped by η upon elements in P. It is readily seen that $N \leq E$ and that E/N is the kernel of the homomorphism of S/N upon T/P which is induced by η. Thus E is the kernel of the homomorphism $\nu\eta$ for ν the natural homomorphism of S upon S/N. Since E is part of R, and since η induces an isomorphism of R/M upon T/P, we infer $E \leq M$ from $E\eta \leq P$. Since η induces clearly an isomorphism of S/E upon T/P, we may apply Theorem 3, (iii) proving that $E = M \cap S$ and $R = M + S$. Thus we have

shown that (i) is a consequence of (iii), and that furthermore the following statement holds true.

COROLLARY 2. *If the subloops M, N, R, S meet the requirements of Theorem 4, and if condition (iii) of Theorem 4 is satisfied by them, then $(M \cap S)/N$ is the kernel of the homomorphism of S/N upon T/P which is induced by η.*

2. Zassenhaus' isomorphism theorem.

LEMMA 1. *If N is a normal subloop of the loop L, then the homomorphism η of L maps N upon a normal subloop $N\eta$ of $L\eta$.*

This is an immediate consequence of the formulas (1), characteristic for normal subloops, which we stated in the introduction.

LEMMA 2.[6] *If U and V are subloops of the loop L, and if W is a subloop of U, then*[7] $(U \cap V) + W = U \cap (V + W)$.

Proof. Since $W \leq U$ and $V \leq V + W$, we have clearly $(U \cap V) + W \leqq U \cap (V + W)$. If x is an element in $U \cap (V + W)$, then x belongs to U and there exist elements v and w in V and W, respectively, such that $x = v + w$. But w belongs, as an element in W, to U; and x belongs to U. Since U is a subloop of L, this implies that v belongs to U too so that v is in $U \cap V$ and x is in $(U \cap V) + W$, proving $U \cap (V + W) \leq (U \cap V) + W$; and this completes the proof of the desired equation.

THEOREM 1. *If R and S are subloops of the loop L, and if M and N are normal subloops of R and S respectively, then*

(a) *$(R \cap N) + M$ is a normal subloop of the subloop $(R \cap S) + M$ of L,*

(b) *$(M \cap S) + (N \cap R)$ is a normal subloop of $R \cap S$,*

(c) *the identity automorphism of L induces an isomorphism of*

$$(R \cap S)/[(M \cap S) + (N \cap R)] \quad \textit{upon} \quad [(R \cap S) + M]/[(R \cap N) + M].$$

Proof.[8] Since M is a normal subloop of R, it is a normal subloop of both $\{(R \cap N), M\}$ and $\{(R \cap S), M\}$, as follows from Theorem 1, (b) of **1.**

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[6] Ore's quadrilateral condition is an almost immediate consequence of this Lemma 2 and of Theorem 1 of **1**; see Ore (1) and Smiley (1), Theorem 5 on p. 784.

[7] This equation is an equality of sets of elements in a loop, not necessarily of subloops. The author is indebted to Mr. R. Good for this improvement of his original version.

[8] The proof is an adaptation of the proof by Zassenhaus (1), p. 50 *et seq.*

Hence it follows from Theorem 1, (a) of 1 that

$$\{(R \cap N), M\} = (R \cap N) + M \quad \text{and} \quad \{(R \cap S), M\} = (R \cap S) + M.$$

Since N is a normal subloop of S, there exists a homomorphism η of S whose kernel is exactly N. Clearly η defines a homomorphism on the subloop $R \cap S$ of S whose kernel is $N \cap R$. Thus $N \cap R$ is a normal subloop of $R \cap S$. The natural homomorphism of R upon R/M maps $R \cap N$ upon $[(R \cap N) + M]/M$; and $R \cap S$ upon $[(R \cap S) + M]/M$; and hence it follows from Lemma 1 that $[(R \cap N) + M]/M$ is a normal subloop of $[(R \cap S) + M]/M$. Consequently there exists a homomorphism v of $[(R \cap S) + M]/M$ whose kernel is just $[(R \cap N) + M]/M$. If ν is the natural homomorphism of $(R \cap S) + M$ upon $[(R \cap S) + M]/M$, then νv is a homomorphism whose kernel is exactly $(R \cap N) + M$, showing the validity of (a).

It is a consequence of Lemma 2 that

$$(R \cap S) \cap [M + (R \cap N)] = (R \cap S \cap M) + (R \cap N) = (S \cap M) + (R \cap N).$$

It is a consequence of (a) that $(R \cap N) + M$ is a normal subloop of the subloop $(R \cap S) + M$ of L. Hence it follows from Theorem 1, (b) of 1 that $(R \cap S) \cap [(R \cap N) + M]$ is a normal subloop of $R \cap S$, since

$$(R \cap S) + M = (R \cap S) + [(R \cap N) + M].$$

This proves our present contention (b). Finally (c) is an immediate consequence of the results already obtained in conjunction with Theorem 1, (c) (or Theorem 3) of 1.

COROLLARY 1. *If R and S are subloops of the loop L, and if M and N are normal subloops of R and S, respectively, then*

$$[(R \cap S) + M]/[(R \cap N) + M] \quad \textit{and} \quad [(R \cap S) + N]/[(S \cap M) + N]$$

are isomorphic quotient loops.

This is an immediate consequence of Theorem 1, (c) and of the symmetry of the hypotheses.

THEOREM 2. *The following properties of the subloops R and S of the loop L and of the normal subloops M and N of R and S, respectively, imply each other.*

(i) $R = (R \cap S) + M$ *and* $S = (R \cap S) + N$.

(ii) *There exists a subloop T of L such that the identity automorphism of L induces both a homomorphism of T upon R/M and a homomorphism of T upon S/N.*

(iii) *There exists a homomorphism η of a loop H into L which induces both a homomorphism of H upon R/M and a homomorphism of H upon S/N.*

Proof. If (i) is satisfied by R, S, M, N, then we put $T = R \cap S$. Since $0 \leq M$ and $R = T + M$, it follows from Theorem 4 of **1** that the identity automorphism of L induces a homomorphism of T upon R/M; and likewise we see that the identity automorphism of L induces a homomorphism of T upon S/N, proving that (ii) is a consequence of (i).

If (ii) holds, then let $H = T$ and denote by η the homomorphism of H into L which is obtained by mapping every element in T upon itself. Thus (iii) is a consequence of (ii).

Assume finally the validity of (iii). Then $R = \{M, H\eta\}$. But M is a normal subloop of R; and thus it follows from Theorem 1, (a) of **1** that $R = M + H\eta$. Clearly $H\eta \leq S$; and thus we find that

$$R = M + H\eta \leq M + (R \cap S) \leq M + R = R \quad \text{or} \quad R = M + (R \cap S);$$

and likewise we verify $S = N + (R \cap S)$, proving that (i) is a consequence of (iii).

THEOREM 3. *The following properties of the subloops R and S of the loop L and of the normal subloops M and N of R and S, respectively, imply each other.*

(i) *There exists a subloop T of L and a normal subloop P of T such that the identity automorphism of L induces both an isomorphism of T/P upon R/M and an isomorphism of T/P upon S/N.*

(ii) *There exists a normal subloop Q of a loop H and a homomorphism η of H into L which induces both an isomorphism of H/Q upon R/M and an isomorphism of H/Q upon S/N.*

Proof. If (i) is true then the validity of (ii) is shown by taking $H = T$, $Q = P$ and η that homomorphism of H into L which maps every element in T upon itself.

If conversely (ii) is satisfied by R, S, M, N, then we put $T = H\eta$ and $P = Q\eta$. It is clear that T and P are subloops of L; and it is a consequence of Lemma 1 that P is a normal subloop of T. Since η induces an isomorphism α of H/Q upon R/M, we have $T \leq R$ and $P \leq M$. Thus the identity automorphism of L induces a homomorphism β of T/P into R/M. We note finally that η induces a homomorphism γ of H/Q upon T/P. Consequently $\alpha = \gamma\beta$; and β must, therefore, be an isomorphism of T/P upon R/M; and likewise

we see that the identity automorphism of L induces an isomorphism of T/P upon S/N, proving that (i) is a consequence of (ii).

THEOREM 4. *The following properties of the subloops R and S of the loop L and of the normal subloops M and N of R and S, respectively, imply each other.*

(i) $R = (R \cap S) + M, S = (R \cap S) + N,$
$M = (R \cap N) + M, N = (S \cap M) + N.$

(ii) $R \cap N = S \cap M$; *and there exists a normal subloop P of a subloop T of L such that the identity automorphism of L induces an isomorphism of T/P upon R/M and an isomorphism of T/P upon S/N.*

Proof. If (i) is satisfied, then $R \cap N \leq M$ and hence $M \cap N \leq R \cap N \leq M \cap N$ or $R \cap N = M \cap N$; and likewise we find that $S \cap M = M \cap N$, proving the first part of condition (ii). Let $T = R \cap S$ and $P = M \cap N$. Then $P = (R \cap N) + (S \cap M)$; and it follows from Theorem 1, (b) that P is a normal subloop of T. From Theorem 1, (c) we infer that the identity automorphism induces an isomorphism of T/P upon R/M, since—by (i)—$R = (R \cap S) + M$ and $M = (R \cap N) + M$; and likewise we see that the identity automorphism of L induces an isomorphism of T/P upon S/N, proving that (ii) is a consequence of (i).

Assume, conversely, the validity of (ii). Then we infer from Theorem 3 of **1** that

$$T = M + T, \quad P = M \cap T \quad \text{and} \quad S = N + T, \quad P = N \cap T.$$

As in the proof of Theorem 2 we see that

$$R = M + T \leq M + (R \cap S) \leq R \quad \text{or} \quad R = M + (R \cap S)$$

and likewise we verify $S = N + (R \cap S)$. From the first part of condition (ii) we deduce that

$$M \leq M + (R \cap N) = M + (S \cap M) \leq M \quad \text{or} \quad M = M + (R \cap N);$$

and in the same fashion we show that $N = N + (S \cap M)$, proving that (i) is a consequence of (ii).

Remark. The impossibility of omitting the first part of condition (ii) may be seen from the following considerations. Suppose that L is an abelian group (= an associative and commutative loop), and that the subgroups M, N, T satisfy: $M \cap T = N \cap T = 0$. Clearly $R = M + T$ is the direct sum of M and T; and $S = N + T$ is the direct sum of N and T. Thus the identity automorphism of L induces an isomorphism of T upon R/M and of T upon

S/N. But clearly $R \cap N$ and $S \cap M$ need not be equal. As an example let L be a free abelian group of rank 2; and let t, m be a basis of L. Put $T = \{t\}$, $M = \{m\}$ and $N = \{m + t\}$. Then $T \cap M = T \cap N = 0$, but

$$R \cap N = (M + T) \cap N = L \cap N = N, \qquad S \cap M = (N + T) \cap M = L \cap M = M \neq N.$$

3. The Theorems of Jordan–Hölder–Schreier–Zassenhaus. Suppose that M is a system of multipliers (= operators) for the elements in the loop L. Then there exists to every element x in L and to every element m in M a uniquely determined element xm in L, subject to the distributive law:

$$(x + y)m = xm + ym \text{ for } x, y \text{ in } L \text{ and } m \text{ in } M.$$

Clearly every element in M induces an endomorphism of L (= homomorphism of L into itself).

A subloop S of L is *M-admissible*, if $SM \leq S$. If S and T are M-admissible subloops of L, and if S is a normal subloop of T, then we infer from the lemma of **1** that M may be considered as a system of multipliers for T/S too.

Suppose now that the system Δ of subloops of L meets the following requirements.

(a) *0 and L belong to Δ.*

(b) *If S and T belong to Δ, then $S \cap T$ and $\{S, T\}$ belong to Δ.*

(c) *Every subloop in Δ is M-admissible.*

The subloops in Δ will be referred to as the *Δ-subloops of L*.

Examples of systems Δ: The system of all M-admissible subloops; the system of all M-admissible normal subloops of L; the system of all M-admissible and M-characteristic subloops of L; the system of all M-admissible and M-characteristic normal subloops of L etc.

Here we say that an endomorphism ϵ of L is an M-endomorphism, if $xm\epsilon = x\epsilon m$ for every x in L and every m in M; and terms like M-characteristic, M-fully-invariant etc. are now self-explanatory. It should be noted, however, that fully invariant subloops need not be normal.[9]

A *Δ-chain of length n* is a system C of Δ-subloops $S(i)$ of L such that

$$S(0) = 0, \; S(i-1) \text{ is a normal subloop of } S(i), \; S(n) = L.$$

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[9] The author is indebted to Professor R. H. Bruck for pointing out this possibility. Bruck (1), Theorem 1, pp. 770-771 shows the existence of a simple loop H containing one and only one maximal subloop $M \neq 0$, $M \neq H$. Clearly M is not normal, but fully invariant.

If C and D are Δ-chains of length m and n respectively, then D is termed a refinement of C, if $m \leq n$ and C is a subset of D. The Δ-chains $C = (\cdots C(i) \cdots)$ and $D = (\cdots D(j) \cdots)$ are said to be isomorphic, if they are of equal length n, and if there exists a permutation: $i \to i'$ of the integers between 0 and n such that $C(i+1)/C(i)$ and $D(i'+1)/D(i')$ are M-isomorphic.

THEOREM. *Any two Δ-chains of L possess isomorphic refinements.*

Proof.[10] If $C(i)$ for $0 \leq i \leq m$ and $D(j)$ for $0 \leq j \leq n$ are two Δ-chains of L, then it follows from Theorem 1 of **2** that

$$[(C(i+1) \cap D(j+1)) + C(i)]/[(C(i+1) \cap D(j)) + C(i)]$$

and

$$[(C(i+1) \cap D(j+1)) + D(j)]/[(C(i) \cap D(j+1)) + D(j)]$$

are M-isomorphic loops. Letting $C^*(in+j) = (C(i+1) \cap D(j)) + C(i)$, $D^*(jm+i) = (C(i) \cap D(j+1)) + D(j)$ and $(in+j)' = jm+i$, the proof is easily completed.

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[10] Cf. Zassenhaus (1), p. 51.